ON THE HARDY-SOBOLEV EQUATION

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ABSTRACT. In this paper we study the problem

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2}u = u^p & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \end{cases} \tag{1}$$

where $\Omega = \mathbb{R}^N$ or $\Omega = B_1$, $N \geq 3$, $p > 1$ and $\lambda < \frac{(N-2)^2}{4}$. Using a suitable map we transform the problem (1) into a another one without the singularity $\frac{1}{|x|^2}$. Then we obtain some bifurcation results from the radial solutions corresponding to some explicit values of λ.

1. Introduction, statement of the main results and idea of the proofs.

In this paper we consider the following problem

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2}u = u^p & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, $p > 1$ and $\lambda < \frac{(N-2)^2}{4}$. We will focus on the case where either $\Omega = \mathbb{R}^N$ or $\Omega = B_1$ and p suitably chosen. By solutions we meen weak solutions, so we will ask that $u \in D^{1,2}\left(\mathbb{R}^N\right)$ where $D^{1,2}\left(\mathbb{R}^N\right) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ such that } |\nabla u| \in L^2(\mathbb{R}^N)\}$ in the first case, or $u \in H_0^1(B_1)$ in the case of the ball. These problems were very studied in the pasts years using both variational or moving plane methods or the finite dimensional reduction of Lyapunov-Schmidt.

In this paper we follow a different approach that will allow us to obtain, among other results, richer multiplicity results of solutions. The main ingredient of our proofs is given by the following map,

$$\mathcal{L}_p : C(0, +\infty) \to C(0, +\infty)$$

defined as

$$v(r) = \mathcal{L}_p(u(r)) = r^a u\left(r^b\right) \quad \text{for } r > 0,\ p > 1 \tag{1.2}$$

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with

$$a = 2\frac{(N-2)(1-\nu_\lambda)}{(p-1)(N-2)(\nu_\lambda-1)+4}, \tag{1.3}$$

and

$$b = \frac{4}{(p-1)(N-2)(\nu_\lambda-1)+4}. \tag{1.4}$$

where

$$\nu_\lambda = \sqrt{1-\frac{4\lambda}{(N-2)^2}}. \tag{1.5}$$

Let $(0,T)\subset \mathbb{R}$ be an interval ($T=+\infty$ is allowed) and $\mathcal{D}^{1,2}((0,T),r^{N-1}dr) = \Big\{u:(0,T)\to\mathbb{R}$ such that $\int_0^T |u'|^2 r^{N-1}dr < +\infty\Big\}$. The next proposition highlights the main properties of the operator $\mathcal{L}_p$.

Proposition 1.1. *Let $p>1$, $\lambda<\frac{(N-2)^2}{4}$ and u be a function satisfying*

$$-u''-\frac{N-1}{r}u'-\frac{\lambda}{r^2}u = u^p \quad \text{in } (0,T) \tag{1.6}$$

with $T\in(0,+\infty]$. Then, we have that $v(r)=\mathcal{L}_p(u(r))$ satisfies

$$-v''-\frac{M-1}{r}v' = A(\lambda,p)v^p \quad \text{in } (0,T^{\frac{1}{b}}) \tag{1.7}$$

where

$$M-1 = \frac{(p+3)(N-2)(\nu_\lambda-1)+4(N-1)}{(p-1)(N-2)(\nu_\lambda-1)+4} \tag{1.8}$$

and

$$A(\lambda,p) = b^2 = \left(\frac{4}{(p-1)(N-2)(\nu_\lambda-1)+4}\right)^2. \tag{1.9}$$

If we choose

$$T=+\infty \text{ when } p=\frac{N+2}{N-2} \tag{1.10}$$

or

$$T=1 \text{ when } 1<p<\frac{N+2}{N-2}, \tag{1.11}$$

then we have that

$$\mathcal{L}_p:\mathcal{D}^{1,2}((0,T),r^{N-1}dr)\to\mathcal{D}^{1,2}((0,T),r^{M-1}dr) \tag{1.12}$$

and

$$||\mathcal{L}_p u||_{\mathcal{D}^{1,2}((0,T),r^{M-1}dr)} = \frac{1}{\nu_\lambda}\int_0^T \left(u'(s)^2-\frac{\lambda}{s^2}u^2(s)\right)s^{N-1}ds \tag{1.13}$$

The previous proposition establishes a one-to-one relationship between the radial solutions to (1.1) and the ODE (1.7). This allows us to find some old and new results about radial solutions to (1.1).
On the other hand we stress that the map $\mathcal{L}_p$ will be used also to establish existence of *nonradial* solutions.
In this paper we analyze two different situations: either

$$p = \frac{N+2}{N-2} \text{ and } \Omega = \mathbb{R}^N, \tag{1.14}$$

or

$$1 < p < \frac{N+2}{N-2} \text{ and } \Omega = B_1. \tag{1.15}$$

Let us start considering $p = \frac{N+2}{N-2}$ so that (1.1) becomes

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = C(\lambda) u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N \\ u \geq 0 \\ u \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{1.16}$$

where $N \geq 3$, $D^{1,2}\left(\mathbb{R}^N\right) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ such that } |\nabla u| \in L^2(\mathbb{R}^N)\}$ and $C(\lambda) = N(N-2)\left(1 - \frac{4\lambda}{(N-2)^2}\right)$ (we have added the constant $C(\lambda)$ just to have a simpler expression of the explicit radial solutions).
Our starting point is the paper [T] of Terracini. For what concerns the radial case, Terracini shows that the unique radial solutions of (1.16) in $D^{1,2}\left(\mathbb{R}^N\right)$ are given by the functions

$$u_{\delta,\lambda}(r) = \frac{r^{\frac{N-2}{2}(\nu_\lambda - 1)} \delta^{\frac{N-2}{2}}}{\left(1 + \delta^2 r^{2\nu_\lambda}\right)^{\frac{N-2}{2}}} \tag{1.17}$$

with ν_λ as in (1.5). Moreover she proved the following result.

Theorem 1.2 (Terracini)**.** *Let $\lambda \in [0, \frac{2N}{N-2})$. Then problem (1.16) has a unique (up to rescaling) solution in $D^{1,2}\left(\mathbb{R}^N\right)$. Moreover there exists $\lambda^* < 0$ such that for $\lambda < \lambda^*$ problem (1.16) admits at least two positive solutions in $D^{1,2}\left(\mathbb{R}^N\right)$ which are distinct modulo rescaling. One is radial while the second is not.*

Another existence result was obtained some years later by Jin, Li and Xu ([JLX]) where the authors proved the existence of singular solutions of the form $u(r,\theta) = r^{\frac{2-N}{2}} g(\theta)$, with (r,θ) polar coordinates in $\mathbb{R}^N$.

Finally, we recall a result by Musso and Wei ([MW]) where it was proved the existence of infinitely many solutions for any $\lambda < 0$. Note that *the energy* of this solutions, namely the quantity $E(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right) - \frac{N-2}{2N}\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}$ is large.
The results in [T] are based on the moving plane method (when $\lambda > 0$) and on the analysis of the radially symmetric case in the phase space.
Using the map $\mathcal{L}_{\frac{N+2}{N-2}}$ we give another proof of some results in [T] in the

radial case. In our opinion this approach is simpler. Actually, as showed in Proposition (1.1), since in this case we have that $M = N$ then the map $\mathcal{L}_{\frac{N+2}{N-2}}$ reduces the study of the radial solutions of (1.16) to the well known problem,

$$\begin{cases} -\Delta U = N(N-2)U^{2^*-1} & \text{in } \mathbb{R}^N \\ U \geq 0 \\ U \in D^{1,2}\left(\mathbb{R}^N\right). \end{cases} \tag{1.18}$$

Solutions of (1.18) have been completely classified in [CGS], where the authors proved that the solutions are given by

$$U_\delta(r) = \frac{\delta^{\frac{N-2}{2}}}{(1+\delta^2 r^2)^{\frac{N-2}{2}}} \tag{1.19}$$

with $\delta > 0$ and they are extremal functions for the well-known Sobolev inequality,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq S \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \tag{1.20}$$

for $u \in D^{1,2}\left(\mathbb{R}^N\right)$ and S the best Sobolev constant.
In this way we derive that the function $u_{\delta,\lambda}$ in (1.17) are the unique radial solutions to (1.16) (see Corollary 3.1) and some inequalities involving the Hardy norm (see Proposition 3.2) (these results were proved in Section 4 in [T] using the phase plane).

As we pointed out, the role of map $\mathcal{L}_{\frac{N+2}{N-2}}$ is not restricted only to the radial case. Indeed it can be used to characterize *all* solutions of the linearized problem at $u_{\delta,\lambda}$, namely

$$\begin{cases} -\Delta v - \frac{\lambda}{|x|^2} v = N(N+2)\nu_\lambda^2 u_{\delta,\lambda}^{\frac{4}{N-2}} v & \text{in } \mathbb{R}^N \\ v \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{1.21}$$

Next result classifies the solution to (1.21),

Lemma 1.3. *Let $\lambda < \frac{(N-2)^2}{4}$ and*

$$\lambda_j = \frac{(N-2)^2}{4}\left(1 - \frac{j(N-2+j)}{N-1}\right), \; j \in \mathbb{N}. \tag{1.22}$$

If $\lambda \neq \lambda_j$ then the space of solutions of (1.21) (with $\delta = 1$) has dimension 1 and it is spanned by

$$Z_\lambda(x) = \frac{|x|^{\frac{N-2}{2}(\nu_\lambda - 1)}\left(1 - |x|^{2\nu_\lambda}\right)}{\left(1 + |x|^{2\nu_\lambda}\right)^{\frac{N}{2}}} \tag{1.23}$$

where ν_λ is as defined in (1.5).
If else $\lambda = \lambda_j$ for some $j = 1, \ldots$ then the space of solutions of (1.21) (with

$\delta = 1$) has dimension $1 + \frac{(N+2j-2)(N+j-3)!}{(N-2)!\,j!}$ and it is spanned by

$$Z_{\lambda_j}(x)\ ,\ Z_{j,i}(x) = \frac{|x|^{\frac{N}{2}\nu_{\lambda_j}-\frac{N-2}{2}}Y_{j,i}(x)}{\left(1+|x|^{2\nu_{\lambda_j}}\right)^{\frac{N}{2}}} \tag{1.24}$$

where $\{Y_{j,i}\}$, $i = 1, \dots, \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}$, form a basis of $\mathbb{Y}_j(\mathbb{R}^N)$, the space of all homogeneous harmonic polynomials of degree j in $\mathbb{R}^N$.

A consequence of the previous result is the computation of the Morse index of $u_\lambda = u_{1,\lambda}$.

Proposition 1.4. *Let $u_\lambda := u_{1,\lambda}$ be the radial solution of (1.16), then its Morse index $m(\lambda)$ is equal to*

$$m(\lambda) = \sum_{\substack{0 \le j < \frac{2-N}{2} + \frac{1}{2}\sqrt{N^2 - \frac{16(N-1)\lambda}{(N-2)^2}} \\ j \ integer}} \frac{(N+2j-2)(N+j-3)!}{(N-2)!\,j!}. \tag{1.25}$$

In particular, we have that the Morse index of u_λ changes as λ crosses the values λ_j and also that $m(\lambda) \to +\infty$ as $\lambda \to -\infty$.

Next aim is to obtain multiplicity results of nonradial solutions as $\lambda < 0$ bifurcating from the radial solution $u_\lambda := u_{1,\lambda}$ in (1.17).

For any $h \in \mathbb{N}$ we let $O(h)$ be the orthogonal group in $\mathbb{R}^h$. Our main result for problem (1.16) is the following (see (3.15) for the definition of the space X).

Theorem 1.5. *Let us fix $j \in \mathbb{N}$ and let λ_j as in (1.22). Then*
i) If j is odd there exists at least a continuum of nonradial weak solutions to (1.16), invariant with respect to $O(N-1)$, bifurcating from the pair $(\lambda_j, u_{\lambda_j})$ in $(-\infty, 0) \times X$.
ii) If j is even there exist at least $\left[\frac{N}{2}\right]$ continua of nonradial weak solutions to (1.16) bifurcating from the pair $(\lambda_j, u_{\lambda_j})$ in $(-\infty, 0) \times X$. The first branch is $O(N-1)$ invariant, the second is $O(N-2) \times O(2)$ invariant, etc.
Moreover all these solutions v_λ satisfy

$$\sup_{x \in \mathbb{R}^N} (1+|x|)^\gamma |v_\lambda(x)| \le C$$

where $\gamma \in \mathbb{R}$ satisfies $\frac{N-2}{2} < \gamma < N-2$.

Remark 1.6. *The solutions of Theorem 1.5 are different from the nonradial one founded in [T]. Indeed, the nonradial solution $\bar u$ in [T] satisfies*

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla \bar u|^2 - \frac{\lambda}{|x|^2}\bar u^2\right)}{\left(\int_{\mathbb{R}^N}|\bar u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} < k^{\frac{2}{N}} S < \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{\frac{N-1}{N}} S \tag{1.26}$$

for some integer k and λ large enough.
On the other hand, since

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u_{\lambda_j}|^2-\frac{\lambda}{|x|^2}u_{\lambda_j}^2\right)}{\left(\int_{\mathbb{R}^N}|u_{\lambda_j}|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}=\left(1-\frac{4\lambda_j}{(N-2)^2}\right)^{\frac{N-1}{N}}S, \tag{1.27}$$

our continuum of solutions does not contain $\bar{u}$, at least in a region "close" to the radial branch u_λ.

The same remark applies to the solutions founded in [MW] since there energy is bigger that nS, for some large integer n.

Remark 1.7. *A consequence of the previous remark is that, for λ close to λ_j and j large, problem (1.16) has at least* three *solutions. One of them is the radial function u_λ in (1.17) and the others are nonradial functions. Moreover, if j is an even number sufficiently large, we have at least $\left[\frac{N}{2}\right]+2$ solutions. This shows that the bifurcation diagram of the solutions to (1.16) is very complicated for $\lambda<0$ and, of course, quite difficult to describe.*

We now consider the case $1<p<\frac{N+2}{N-2}$ and the problem

$$\begin{cases} -\Delta u-\frac{\lambda}{|x|^2}u=u^p & \text{in } B_1\\ u>0 & \text{in } B_1\\ u\in H_0^1(B_1) \end{cases} \tag{1.28}$$

where B_1 is the unit ball in $\mathbb{R}^N$. A complete description of (1.28) for $\lambda\ge 0$ was given in [CG] where the authors proved that, problem (1.28) admits a unique solution which is radial. We are not aware of any result instead for $\lambda<0$. Anyway, since the problem is subcritical then the existence of a radial solution is a straightforward consequence of the Mountain Pass Theorem. Likewise to the previous case, Proposition (1.1) shows that the map $\mathcal{L}_p$ provides an equivalence between the radial solutions to (1.28) and the solutions of

$$\begin{cases} -v''-\frac{M-1}{r}v'=A(\lambda,p)v^p & \text{in } (0,1)\\ v>0 & \text{in } (0,1)\\ v'(0)=v(1)=0 \end{cases} \tag{1.29}$$

with M and $A(\lambda,p)$ as in (1.8), (1.9) (see Section 4 for a discussion about the boundary conditions). It is known that this problem has a unique solution that we call v_λ. Then we get the following result:

Theorem 1.8. *Let $\lambda<\frac{(N-2)^2}{4}$ and $1<p<\frac{N+2}{N-2}$. Then the problem (1.28) admits only one radial solution $u_\lambda(r)$. Moreover*

$$r^{\frac{(N-2)}{2}(1-\nu_\lambda)}u_\lambda(r)\to C \qquad \text{as } r\to 0^+ \tag{1.30}$$

where $C>0$.

This result extends the uniqueness result of [CG] to the case $\lambda < 0$ and shows that radial solutions to (1.28) satisfy $u_\lambda(0) = 0$ for $\lambda < 0$ differently from the case of $\lambda \geq 0$ where $u_\lambda \notin L^\infty(B_1)$. Then the monotonicity properties of the Gidas, Ni and Nirenberg theorem cannot hold when $\lambda < 0$.
Once we have this branch of radial solution u_λ for any $\lambda < 0$ we can look for nonradial solutions that arise by bifurcation. The strategy to obtain multiplicity results for $\lambda < 0$ is the same as in the case $p = \frac{N+2}{N-2}$. First we prove non degeneracy of u_λ in the space of radial function. Then we characterize the values of λ for which the linearized operator at the radial solution u_λ is non invertible and we compute the change of the Morse index of the radial solutions at these points. These values of λ are related to a weighted eigenvalue for problem (1.29). To this end we let $v_\lambda \in H^1((0,1), r^{M-1}dr)$ be the unique solution to (1.29) and set

$$\Lambda(\lambda) = \inf_{w \in H^1((0,1), r^{M-1}dr),\ w(1)=0} \frac{\int_0^1 \left(|w'|^2 - pA(\lambda,p)v_\lambda^{p-1}w^2\right) r^{M-1}}{\int_0^1 w^2 r^{M-3}}. \tag{1.31}$$

The infimum $\Lambda(\lambda)$ is well defined by the Hardy inequality but it is not clear if it is achieved. Indeed the embedding of $H^1((0,1), r^{M-1}dr) \hookrightarrow L^2((0,1), r^{M-3}dr)$ is not compact. However the crucial information that the infimum (1.31) is strictly negative implies that it is attained. This is proved in Proposition 5.8 in the Appendix in a more general case and relies on a careful study of some weighted problem given in [GGN2], Section 2.

Now we can state the following result,

Theorem 1.9. *For any $j \in \mathbb{N}$, $j \geq 1$ there exists a value λ_j that satisfies*

$$\left(\frac{4}{(p-1)\left(2 - N + \sqrt{(N-2)^2 - 4\lambda}\right) + 4}\right)^2 j(N-2+j) = -\Lambda(\lambda) \tag{1.32}$$

and an interval $I_j \subset (-\infty, 0)$ such that $\lambda_j \in I_j$ and that a nonradial bifurcation occurs at (λ, u_λ) for $\lambda \in I_j$.
Moreover, if j is even there exist at least $\left[\frac{N}{2}\right]$ continua of nonradial solutions to (1.28) bifurcating from (λ, u_λ) for $\lambda \in I_j$. The first branch is $O(N-1)$ invariant, the second is $O(N-2) \times O(2)$ invariant, etc.

The paper is organized as follows: in Section 2 we show the main properties of the map $\mathcal{L}_p$. In Section 3 we consider the case $p = \frac{N+2}{N-2}$ and in Section 4 the sub critical case $1 < p < \frac{N+2}{N-2}$. In the Appendix we prove some technical result.

2. Main properties of the map $\mathcal{L}_p$

In this section we give the proof of Proposition 1.1.

Proof of Proposition 1.1. Let $u(r)$ be a solution of (1.6). Then a straightforward computation shows that $v(r)=\mathcal{L}_p(u(r))$ is a solution to (1.7) with M and $A(\lambda,p)$ satisfying (1.8) and (1.9) respectively.
Now let us show (1.13). We just consider the case $p=\frac{N+2}{N-2}$ and $T=+\infty$ (the subcritical case $1<p<\frac{N+2}{N-2}$ is similar and easier).
Note that in this case we have $a=\frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}$, $b=\frac{1}{\nu_\lambda}$ and

$$v(r)=r^{\frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}}u\left(r^{\frac{1}{\nu_\lambda}}\right)\quad\text{for } r>0. \tag{2.1}$$

First of all we observe that since $u\in\mathcal{D}^{1,2}((0,+\infty),r^{N-1}dr)$ then $\int_0^{+\infty}u(s)^{\frac{2N}{N-2}}s^{N-1}<+\infty$ and so there exist sequences $\delta_n\to 0$ and $M_n\to+\infty$ such that

$$u(\delta_n)\delta_n^{\frac{N-2}{2}}\to 0\quad\text{and}\quad u(M_n)M_n^{\frac{N-2}{2}}\to 0.$$

By (2.1) we derive that

$$\frac{r^{\frac{N}{\nu_\lambda}-N}}{\nu_\lambda^2}\left(u'\left(r^{\frac{1}{\nu_\lambda}}\right)\right)^2=\frac{(N-2)^2(1-\nu_\lambda)^2}{4\nu_\lambda^2}\frac{v^2(r)}{r^2}-\frac{(N-2)(1-\nu_\lambda)}{\nu_\lambda}\frac{v(r)v'(r)}{r}+\big(v'(r)\big)^2$$

Choosing $\varepsilon_n=\delta_n^{\nu_\lambda}$ and $R_n=M_n^{\nu_\lambda}$ and integrating we get

$$\begin{aligned}\frac{1}{\nu_\lambda^2}\int_{\varepsilon_n}^{R_n}\left(u'\left(r^{\frac{1}{\nu_\lambda}}\right)\right)^2 r^{\frac{N}{\nu_\lambda}-1}&=\frac{(N-2)^2(1-\nu_\lambda)^2}{4\nu_\lambda^2}\int_{\varepsilon_n}^{R_n}v^2(r)r^{N-3}\\ -\frac{(N-2)(1-\nu_\lambda)}{\nu_\lambda}\int_{\varepsilon_n}^{R_n}v(r)v'(r)r^{N-2}&+\int_{\varepsilon_n}^{R_n}\big(v'(r)\big)^2r^{N-1}.\end{aligned} \tag{2.2}$$

Then we have, using again (2.1)

$$\int_{\varepsilon_n}^{R_n}v(r)v'(r)r^{N-2}=\frac{1}{2}r^{N-2}v^2(r)\Big|_{\varepsilon_n}^{R_n}-\frac{N-2}{2}\int_{\varepsilon_n}^{R_n}v^2(r)r^{N-3}=$$
$$\frac{1}{2}\left(u(M_n)M_n^{\frac{N-2}{2}}\right)^2-\frac{1}{2}\left(u(\delta_n)\delta_n^{\frac{N-2}{2}}\right)^2-\frac{N-2}{2}\int_{\varepsilon_n}^{R_n}v^2(r)r^{N-3}.$$

Hence, by the choice of ε_n and R_n we deduce that

$$\int_0^{+\infty}v(r)v'(r)r^{N-2}=-\frac{N-2}{2}\int_0^{+\infty}v^2(r)r^{N-3}.$$

So (2.2) becomes

$$\frac{1}{\nu_\lambda}\int_0^{+\infty}\left(u'(s)\right)^2 s^{N-1} = \frac{(N-2)^2(1-\nu_\lambda)}{4\nu_\lambda^2}(1+\nu_\lambda)\int_0^{+\infty} v^2(r)r^{N-3} +$$

$$\int_0^{+\infty}\left(v'(r)\right)^2 r^{N-1}(\text{recalling the definition of } \nu_\lambda) =$$

$$\frac{\lambda}{\nu_\lambda}\int_0^{+\infty} u^2(s)s^{N-3} + \int_0^{+\infty}\left(v'(r)\right)^2 r^{N-1},$$

which gives the claim. □

3. The critical case $p = \frac{N+2}{N-2}$

3.1. **Basic properties and the main inequality.** In this section we consider problem (1.16). First let us observe that if we put $p = \frac{N+2}{N-2}$ in (1.2)-(1.5) we get

$$a = \frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}, \tag{3.1}$$

and

$$b = \frac{1}{\nu_\lambda}. \tag{3.2}$$

Moreover, if $u \in \mathcal{D}^{1,2}((0,+\infty), r^{N-1}dr)$ satisfies in weak sense

$$-u'' - \frac{N-1}{r}u' - \frac{\lambda}{r^2}u = C(\lambda)u^{2^*-1} \quad \text{in } (0,+\infty) \tag{3.3}$$

where $C(\lambda) = N(N-2)\nu_\lambda^2$ and ν_λ as in (1.5), then Proposition 1.1 shows that $v \in \mathcal{D}^{1,2}((0,+\infty), r^{N-1}dr)$ weakly solves

$$-v'' - \frac{N-1}{r}v' = N(N-2)v^{2^*-1} \quad \text{in } (0,+\infty). \tag{3.4}$$

Corollary 3.1. *All the radial solutions in $D^{1,2}(\mathbb{R}^N)$ of (1.16) are given by the functions $u_{\delta,\lambda}(r)$ in (1.17).*

Proof. It follows directly by (3.3) and (3.4). Since all solution to (3.4) are given by $v_\delta(r) = \frac{\delta^{\frac{N-2}{2}}}{(1+\delta^2 r^2)^{\frac{N-2}{2}}}$, by the definition of $\mathcal{L}_p$ we deduce that

$$u_{\delta,\lambda}(r) = r^{-a}v_\delta\left(r^{\frac{1}{b}}\right) = r^{\frac{(N-2)(\nu_\lambda-1)}{2\nu_\lambda}}v_\delta\left(r^{\nu_\lambda}\right) = \frac{r^{\frac{(N-2)(\nu_\lambda-1)}{2\nu_\lambda}}\delta^{\frac{N-2}{2}}}{(1+\delta^2 r^{2\nu_\lambda})^{\frac{N-2}{2}}} \tag{3.5}$$

which gives the claim. □

Now we prove an interesting inequality. We remark that, in the case $\lambda \geq 0$ this is basically contained in [T]. If $\lambda < 0$ we do not find any references although this can be shown using (for example) the concentration-compactness principle of Lions. Anyway, we think that the interest of the next proposition is in its proof, which reduces the Hardy inequality to the classical Sobolev imbedding.

Proposition 3.2. *Let $\lambda < \frac{(N-2)^2}{4}$. Then we have that for any radial function $u \in D^{1,2}\left(\mathbb{R}^N\right)$*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) \geq \left(1 - \frac{4\lambda}{(N-2)^2} \right)^{\frac{N-1}{N}} S \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}} \tag{3.6}$$

where S is the best Sobolev constant. Moreover the previous inequality is achieved only for $u(r) = u_{\delta,\lambda}(r)$.
If $\lambda > 0$ then (3.6) holds for any $u \in D^{1,2}\left(\mathbb{R}^N\right)$.

Proof of Proposition 3.2. Let v be as in (2.1). Then by (1.13) we get

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) = \nu_\lambda \int_{\mathbb{R}^N} |\nabla v(|x|)|^2 \geq \nu_\lambda S \left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}$$

$$= \nu_\lambda S \left(\omega_N \int_0^{+\infty} |v(r)|^{\frac{2N}{N-2}} r^{N-1} \right)^{\frac{N-2}{N}} = \nu_\lambda^{2\frac{N-1}{N}} S \left(\omega_N \int_0^{+\infty} |u(s)|^{\frac{2N}{N-2}} s^{N-1} \right)^{\frac{N-2}{N}} =$$

$$\left(1 - \frac{4\lambda}{(N-2)^2} \right)^{\frac{N-1}{N}} S \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}$$

which gives the claim. Note that the previous inequality becomes an identity if and only if $\int_{\mathbb{R}^N} |\nabla v(|x|)|^2 = S \left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}$. It is well known (see for example [CGS]) that this implies $v(x) = \frac{\delta^{\frac{N-2}{2}}}{(1+\delta^2 r^2)^{\frac{N-2}{2}}}$ for some positive δ. Recalling that (see Corollary 3.1) $u(r) = r^{-\frac{(N-2)(1-\nu_\lambda)}{2}} v\left(r^{\nu_\lambda}\right)$ we have the uniqueness of the minimizer.
Let us show that if $\lambda > 0$ then (3.6) holds for any $u \in D^{1,2}\left(\mathbb{R}^N\right)$. This follows by the classical spherical rearrangement theory. Indeed, denoting by $u^* = u^*(|x|)$ the classical Schwartz rearrangement we have that $\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2$, $\int_{\mathbb{R}^N} |u^*|^{\frac{2N}{N-2}} = \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}$ and $\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \leq \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^2}$.

Hence, if $\lambda > 0$, we get

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u^*|^2 - \frac{\lambda}{|x|^2}|u^*|^2\right)}{\left(\int_{\mathbb{R}^N}|u^*|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} \le \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right)}{\left(\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}$$

which implies the claim. $\square$

3.2. **The linearized operator.** In this section we linearize problem (1.16) at the radial solution $u_\lambda := u_{1,\lambda}$ given in (1.17) and we look for the degeneracy points. This is equivalent to find nontrivial solutions for the linearized problem (1.21). Using (1.17) we rewrite (1.21) as follows

$$\begin{cases} -\Delta v - \frac{\lambda}{|x|^2}v = N(N+2)\nu_\lambda^2\frac{|x|^{2(\nu_\lambda-1)}}{\left(1+|x|^{2\nu_\lambda}\right)^2}v & \text{in } \mathbb{R}^N \\ v \in D^{1,2}\left(\mathbb{R}^N\right). \end{cases} \tag{3.7}$$

We solve (3.7) using the decomposition along the spherical harmonic functions and we write

$$v(r,\theta) = \sum_{j=0}^{\infty}\psi_j(r)Y_j(\theta), \qquad \text{where} \qquad r = |x|\ ,\ \theta = \frac{x}{|x|} \in S^{N-1}$$

and

$$\psi_j(r) = \int_{S^{N-1}} V(r,\theta)Y_j(\theta)d\theta.$$

Here $Y_j(\theta)$ denotes the j-th spherical harmonics, i.e. it satisfies

$$-\Delta_{S^{N-1}}Y_j = \mu_j Y_j$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} with the standard metric and μ_j is the j-th eigenvalue of $-\Delta_{S^{N-1}}$. It is known that

$$\mu_j = j(N-2+j), \qquad j = 0,1,2,\dots \tag{3.8}$$

whose multiplicity is

$$\frac{(N+2j-2)(N+j-3)!}{(N-2)!\,j!} \tag{3.9}$$

and that

$$Ker\left(\Delta_{S^{N-1}} + \mu_j\right) = \mathbb{Y}_j(\mathbb{R}^N)\,|_{S^{N-1}}$$

and $\mathbb{Y}_j(\mathbb{R}^N)$ is the space of all homogeneous harmonic polynomials of degree j in $\mathbb{R}^N$. The function v is a weak solution of (3.7) if and only if $\psi_j(r)$ is a weak solution of

$$\begin{cases} -\psi_j''(r) - \frac{N-1}{r}\psi_j'(r) + \frac{\mu_j-\lambda}{r^2}\psi_j(r) = N(N+2)\nu_\lambda^2\frac{r^{2(\nu_\lambda-1)}}{\left(1+r^{2\nu_\lambda}\right)^2}\psi_j & \text{in } (0,\infty) \\ \psi_j'(0) = 0 \text{ if } j=0 \quad \text{ and } \psi_j(0) = 0 \text{ if } j \ge 1 \\ \psi_j \in D^{1,2}((0,+\infty), r^{N-1}dr). \end{cases} \tag{3.10}$$

Letting, as in (1.2),

$$\hat{\psi}_j(r) = r^{\frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}} \psi_j\left(r^{\frac{1}{\nu_\lambda}}\right),$$

we have that $\hat{\psi}_j$ weakly solves

$$\begin{cases} -\hat{\psi}_j''(r) - \frac{N-1}{r}\hat{\psi}_j'(r) + \frac{\mu_j}{\nu_\lambda^2 r^2}\hat{\psi}_j(r) = N(N+2)\frac{1}{(1+r^2)^2}\hat{\psi}_j \quad \text{in } (0,\infty) \\ \hat{\psi}_j \in D^{1,2}((0,+\infty), r^{N-1}dr). \end{cases} \tag{3.11}$$

Problem (3.11) is well known: it comes from the linearization of the solution U_1 in (1.19) of problem (1.18). This equation has a nontrivial solution (since $\frac{\mu_j}{\nu_\lambda^2} \geq 0$) if and only if one of the following holds

i) $\frac{\mu_j}{\nu_\lambda^2} = 0$

ii) $\frac{\mu_j}{\nu_\lambda^2} = N-1$.

Case i) corresponds to the radial degeneracy, i.e. $j = 0$. Equation (3.11) has the solution $\hat{\psi}_0(r) = \frac{1-r^2}{(1+r^2)^{\frac{N}{2}}}$ and turning back to (3.10) we get

$$\psi_0(r) = \frac{r^{\frac{N-2}{2}(\nu_\lambda - 1)}\left(1 - r^{2\nu_\lambda}\right)}{\left(1 + r^{2\nu_\lambda}\right)^{\frac{N}{2}}}$$

which is a solution to (3.10) for any $\lambda < \frac{(N-2)^2}{4}$. This proves (1.23).
When $\frac{\mu_j}{\nu_\lambda^2} = N-1$ then equation (3.11) has the solution $\hat{\psi}_j(r) = \frac{r}{(1+r^2)^{\frac{N}{2}}}$. Turning back to (3.10) we get that (3.10) has the solution

$$\psi_j(r) = \frac{r^{\frac{N-2}{2}(\nu_\lambda - 1) + \nu_\lambda}}{\left(1 + r^{2\nu_\lambda}\right)^{\frac{N}{2}}}$$

when ii) is satisfied. Then ii) implies that (3.10) has the solution $\psi_j(r)$ if and only if $\lambda = \lambda_j$

$$\lambda_j = \frac{(N-2)^2}{4}\left(1 - \frac{\mu_j}{N-1}\right)$$

as in (1.22). This proves (1.24) and finishes the proof of Lemma 1.3.

A first consequence of Lemma 1.3 is the computation of the Morse index of the solution u_λ given in Proposition (1.4).

Proof of Proposition (1.4). As shown in the Appendix (Corollary 5.7), the Morse index of the radial solution u_λ is given by the number of negative values Λ_i such that the problem

$$\begin{cases} -\Delta w - \frac{\lambda}{|x|^2} w - N(N+2)\nu_\lambda^2 \frac{|x|^{2\nu_\lambda}}{\left(1+|x|^{2\nu_\lambda}\right)^2} w = \frac{\Lambda}{|x|^2} w \quad \text{in } \mathbb{R}^N \\ w \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{3.12}$$

admits a weak solution w_i, counted with their multiplicity. We denote by w_i the solution of (3.12) related to a negative value Λ_i. We argue as before setting $w_{i,j}(r)=\int_{S^1} w_i(r,\theta)Y_j(\theta)\,d\theta$ and $\widehat{w}_{i,j}(r)=r^{\frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}}w_{i,j}(r^{\frac{1}{\nu_\lambda}})$. Then $\widehat{w}_{i,j}(r)$ weakly satisfies

$$\begin{cases}-\widehat{w}''_{i,j}(r)-\frac{N-1}{r}\hat{w}'_{i,j}(r)-N(N+2)\frac{1}{(1+r^2)^2}\hat{w}_{i,j}=\frac{\Lambda_i-\mu_j}{\nu_\lambda^2 r^2}\hat{w}_{i,j}(r) & \text{in } (0,\infty)\\ \hat{w}_{i,j}\in D^{1,2}((0,+\infty),r^{N-1}dr).\end{cases} \tag{3.13}$$

Since the problem

$$\begin{cases}-\eta''(r)-\frac{N-1}{r}\eta'(r)-N(N+2)\frac{1}{(1+r^2)^2}\eta=\frac{\nu}{r^2}\eta & \text{in } (0,\infty)\\ \eta\in D^{1,2}((0,+\infty),r^{N-1}dr)\end{cases}$$

admits only one negative eigenvalue which is $1-N$, then we derive that equation (3.13) has a nontrivial solution corresponding to a $\Lambda_i<0$, if and only if

$$1-N=\frac{\Lambda_i-\mu_j}{\nu_\lambda^2}.$$

So we have that the indexes j which contribute to the Morse index of the solution u_λ are those that satisfy $\Lambda_i=\nu_\lambda^2(1-N)+\mu_j<0$ and this implies, recalling the value of μ_j given in (3.8), $j<\frac{2-N}{2}+\frac{1}{2}\sqrt{N^2-\frac{16(N-1)\lambda}{(N-2)^2}}$. Finally, using that the dimension of the eigenspace of the Laplace-Beltrami operator on S^{N-1} related to μ_j is given in (3.9), (1.25) follows. □

Remark 3.3. *Reasoning as in the proof of the previous corollary it is easy to see that any eigenfunction of* (3.12) *corresponding to an eigenvalue* $\Lambda<0$ *can be written in the following way*

$$w(r,\theta)=r^{\frac{(N-2)}{2}(\nu_\lambda-1)}\frac{r^{\nu_\lambda}}{(1+r^{2\nu_\lambda})^{\frac{N}{2}}}Y_j(\theta)$$

where $Y_j(\theta)$ *is a spherical harmonic related to the eigenvalue* μ_j.

3.3. **The bifurcation result.** In this section we will start the proof Theorem 1.5 using the bifurcation theory.
To this end let us give some definitions. Let $\gamma>0$ be such that $\frac{N-2}{2}<\gamma<N-2$. For every $g\in L^\infty(\mathbb{R}^N)$ we define the weighted norm

$$\|g\|_\gamma:=\sup_{x\in\mathbb{R}^N}(1+|x|)^\gamma|g(x)| \tag{3.14}$$

and the space $L^\infty_\gamma(\mathbb{R}^N):=\{g\in L^\infty(\mathbb{R}^N)$ such that $\exists C>0$ and $\|g\|_\gamma<C\}$. Set

$$X=D^{1,2}(\mathbb{R}^N)\cap L^\infty_\gamma(\mathbb{R}^N). \tag{3.15}$$

X is a Banach space with the norm

$$\|g\|_X:=\max\{\|g\|_{1,2},\|g\|_\gamma\} \tag{3.16}$$

where $\|g\|_{1,2}$ denotes the usual norm in $D^{1,2}(\mathbb{R}^N)$, i.e. $\|g\|_{1,2} = \left(\int_{\mathbb{R}^N} |\nabla g|^2\, dx\right)^{\frac{1}{2}}$. To apply the standard bifurcation theory we have to define a compact operator T from $(-\infty, 0) \times X$ into X and to compute its Leray Schauder degree in 0 in a suitable neighborhood of the radial solution (λ, u_λ), at least for the values $\lambda \neq \lambda_j$. This seems difficult since the linearized operator (see (1.21)) is not invertible due to the radial degeneracy of u_λ for every λ proved in Lemma 1.3. To this end we define

$$K_\lambda := \left\{ v \in D^{1,2}(\mathbb{R}^N) \text{ such that } \int\limits_{\mathbb{R}^N} u_\lambda^{2^*-2} v Z_\lambda \, dx = 0 \right\}$$

with Z_λ as defined in (1.23). Since $u_\lambda^{2^*-2} \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $v,\ Z_\lambda \in L^{2^*}(\mathbb{R}^N)$, we have that K_λ is a linear closed subspace of $D^{1,2}(\mathbb{R}^N)$. We let P_λ be orthogonal the projection of $D^{1,2}(\mathbb{R}^N)$ on K_λ.
Now we define the operator $T(\lambda, v) : (-\infty, 0) \times X \to K_\lambda \cap X$ as

$$T(\lambda, v) = P_\lambda \left(\left(-\Delta - \frac{\lambda}{|x|^2} I \right)^{-1} \left(C(\lambda) (v^+)^{2^*-1} \right) \right) \tag{3.17}$$

and look for zeros of the operator $I - T(\lambda, v)$. A function $v \in X$ is a zero of $I - T(\lambda, v)$ if $v \in K_\lambda \cap X$ and v is a weak solution of

$$-\Delta v - \frac{\lambda}{|x|^2} v - C(\lambda) v^{2^*-1} = L C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} Z(x) \quad \text{in } \mathbb{R}^N \tag{3.18}$$

where $L = L(v) \in \mathbb{R}$ (L is the Lagrange multiplier). The final step will be to show that $L = 0$ so that v is indeed a weak solution of (1.16). This will be done in Section 3.4.
Before proving our bifurcation result we need some technical results.

Lemma 3.4. *The operator $T(\lambda, v)$ is well defined from $(-\infty, 0) \times X$ into $K_\lambda \cap X$.*

Proof. It is enough prove that the operator

$$\widetilde{T}(\lambda, v) = \left(-\Delta - \frac{\lambda}{|x|^2} I \right)^{-1} \left(C(\lambda) (v^+)^{2^*-1} \right) \tag{3.19}$$

is well defined from $(-\infty, 0) \times X$ in X.
Since $(v^+)^{2^*-1} \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ there exists a unique $g \in D^{1,2}(\mathbb{R}^N)$ such that $g = \widetilde{T}(\lambda, v)$, see Lemma 5.2 in the Appendix, i.e. g is a weak solution to

$$-\Delta g - \frac{\lambda}{|x|^2} g = C(\lambda) (v^+)^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

Then, the comparison theorem for functions in $D^{1,2}(\mathbb{R}^N)$, yields

$$|g(x)| \le C |w(x)| \quad \text{a. e. in } \mathbb{R}^N$$

where w is the unique weak solution to

$$-\Delta w - \frac{\lambda}{|x|^2} w = \frac{1}{(1+|x|)^{\gamma\frac{N+2}{N-2}}} \quad \text{in } \mathbb{R}^N. \tag{3.20}$$

We are going to prove that

$$(1+r)^\gamma\, w(r) \le C. \tag{3.21}$$

To do this let $\bar{w}(r) = r^k w(r)$, where $k = \frac{N-2}{2}(1-\nu_\lambda)$ and ν_λ is as defined in (1.5). The function $\bar{w}$ weakly satisfies

$$-\bar{w}'' - \frac{N-1-2k}{r}\bar{w}' = \frac{r^k}{(1+r)^{\gamma\frac{N+2}{N-2}}} \quad \text{in } (0,+\infty).$$

Integrating we get

$$-r^{N-1-2k}\bar{w}'(r) = C + \int_{r_0}^{r} \frac{s^{N-1-k}}{(1+s)^{\gamma\frac{N+2}{N-2}}}\, ds$$

for any $r_0 > 0$. Consequently $-\bar{w}'(r) \le C r^{1-N+2k} + C r^{1-N+2k+N-k-\gamma\frac{N+2}{N-2}}$ (we are assuming that $N-1-k-\gamma\frac{N+2}{N-2} \neq -1$; the case $N-k-\gamma\frac{N+2}{N-2} = 0$ follows in a very similar way).
Since $w \in D^{1,2}(\mathbb{R}^N)$, from Ni's radial Lemma (see [Ni]) we know that $w(r) \le C r^{\frac{2-N}{2}}$ for any r, so that $\bar{w}(r) \le C r^{k-\frac{N-2}{2}} = C r^{-\frac{N-2}{2}\nu_\lambda}$. Then $\bar{w}(r) \to 0$ as $r \to +\infty$. Integrating $\bar{w}'(r)$ from r to $+\infty$ yelds

$$\bar{w}(r) \le C r^{2-N+2k} + C r^{2+k-\gamma\frac{N+2}{N-2}}.$$

This implies that, since by assumption $\frac{N-2}{2} < \gamma < N-2$ and $k < 0$ for any $\lambda < 0$,

$$(1+r)^\gamma w(r) \le C r^{\gamma+2-N+k} + C r^{\gamma+2-\gamma\frac{N+2}{N-2}} \le C \tag{3.22}$$

for r large enough.
To finish the proof of (3.21) we need to prove that $|w(x)|$ is bounded in a neighborhood of the origin. To this end we set

$$\widetilde{w}(r) = \frac{1}{r^{N-2k-2}}\bar{w}\left(\frac{1}{r}\right).$$

The function $\widetilde{w}$ is the Kelvin transform of $\bar{w}$ and so it satisfies

$$-\widetilde{w}'' - \frac{N-1-2k}{r}\widetilde{w}' = \frac{1}{r^{N-2k+2}} \frac{r^{-k}}{(1+\frac{1}{r})^{\gamma\frac{N+2}{N-2}}} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Reasoning as before and integrating from r_0 to r we get

$$-r^{N-1-2k}\widetilde{w}'(r) \le C + C\int_{r_0}^{r} s^{-3-k} ds$$

and, assuming $-3-k>-1$,(observe that the case $-3-k<-1$ is easier and the case $-3-k=-1$ follows reasoning as in the first case)

$$-\widetilde{w}'(r)\le Cr^{1-N+2k}+Cr^{1-N+2k-2-k}$$

for r large enough. Then, using that $w(r)\to 0$ as $r\to+\infty$,

$$\widetilde{w}(r)\le Cr^{2-N+2k}+Cr^{-N+k}$$

for r large enough. This implies that

$$\bar{w}(r)=\frac{1}{r^{N-2-2k}}\widetilde{w}\left(\frac{1}{r}\right)\le Cr^{2+k}+C$$

for r small enough. Finally, using that $w(r)=r^{-k}\bar{w}(r)$ we have that

$$w(r)\le C\begin{cases} r^{-k}+r^2 & \text{if } k<-2\\ r^2(1-\log r) & \text{if } k=-2\\ r^{-k} & \text{if } -2<k<0\end{cases} \quad \text{for } r \text{ small enough.} \tag{3.23}$$

Estimates (3.22) and (3.23) imply that

$$\sup_{x\in\mathbb{R}^N}(1+|x|)^{\gamma}|w(x)|\le C$$

so that w and hence g belong to $L^\infty_\gamma(\mathbb{R}^N)$ concluding the proof. □

Proposition 3.5. *We have that:*

i) *the operator $T(\lambda,v):(-\infty,0)\times X\to K_\lambda\cap X$ defined in (3.17) is continuous with respect to λ and it is compact from X into $K_\lambda\cap X$ for any $\lambda\in(-\infty,0)$ fixed;*

ii) *the linearized operator $I-T'_v(\lambda,u_\lambda)I$ is invertible for any $\lambda\neq\lambda_j$, where λ_j are as defined in (1.22).*

Proof. Let us prove $i)$. The operator $T(\lambda,v)$ is clearly continuous with respect to λ. As in the proof of Lemma 3.4, we will prove that the operator $\widetilde{T}$, defined in (3.19) is compact from X into X for every λ fixed. This implies in turn that T is compact for every λ fixed. To this end let v_n be a sequence in X such that $\|v_n\|_X\le C$ and let $g_n=\widetilde{T}(\lambda,v_n)$. Then $g_n\in X$ and by Lemma 3.4 is a weak solution to

$$-\Delta g_n-\frac{\lambda}{|x|^2}g_n=C(\lambda)(v_n^+)^{2^*-1}. \tag{3.24}$$

Since v_n is bounded in X then $|v_n(x)|\le C(1+|x|)^{-\gamma}$ and v_n is uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ then, up to a subsequence, $v_n\rightharpoonup\bar{v}$ weakly in $D^{1,2}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. Multiplying (3.24) by g_n and integrating we get

$$\int\limits_{\mathbb{R}^N}|\nabla g_n|^2\,dx-\int\limits_{\mathbb{R}^N}\frac{\lambda}{|x|^2}g_n^2=C(\lambda)\int\limits_{\mathbb{R}^N}(v_n^+)^{2^*-1}g_n\,dx. \tag{3.25}$$

Then the Hardy and Sobolev inequalities imply that

$$c_\lambda \int\limits_{\mathbb{R}^N} |\nabla g_n|^2\,dx \le C\|v_n^{2^*-1}\|_{\frac{2N}{N+2}}\|g_n\|_{1,2}$$

where c_λ is as in Lemma 5.1 and $\|\cdot\|_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$. Then

$$\|g_n\|_{1,2} \le C$$

so that, up to a subsequence $g_n \rightharpoonup \bar g$ weakly in $D^{1,2}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. Passing to the limit in (3.24), we get that $\bar g$ is a weak solution of

$$-\Delta \bar g - \frac{\lambda}{|x|^2}\bar g = C(\lambda)(\bar v^+)^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

Moreover, reasoning exactly as in the proof of Lemma 3.4 we get also $|g_n(x)| \le C(1+|x|)^{-\gamma}$ for any n. This estimate allow us to pass to the limit in (3.25) getting that

$$\int\limits_{\mathbb{R}^N} |\nabla g_n|^2\,dx - \int\limits_{\mathbb{R}^N} \frac{\lambda}{|x|^2} g_n^2 = C(\lambda)\int\limits_{\mathbb{R}^N} (v_n^+)^{2^*-1} g_n\,dx \to$$

$$C(\lambda)\int\limits_{\mathbb{R}^N} (\bar v^+)^{2^*-1}\bar g\,dx = \int\limits_{\mathbb{R}^N} |\nabla \bar g|^2\,dx - \int\limits_{\mathbb{R}^N} \frac{\lambda}{|x|^2}\bar g^2\,dx.$$

By Lemma 5.1 this implies that $g_n \to g$ strongly in $D^{1,2}(\mathbb{R}^N)$. To finish the proof we need to show that $\|g_n - g\|_\gamma < \varepsilon$ if n is large enough. To this end observe that $g_n - g$ weakly solves

$$-\Delta(g_n - \bar g) - \frac{\lambda}{|x|^2}(g_n - \bar g) = C(\lambda)\left((v_n^+)^{2^*-1} - (\bar v^+)^{2^*-1}\right) \text{ in } \mathbb{R}^N$$

and, since v_n and $\bar v$ are uniformly bounded in $L^\infty_\gamma(\mathbb{R}^N)$ then, as in Lemma 3.4 we have that $|g_n - \bar g| \le Cw$, where w is defined in (3.20). Then, from (3.22) we have that there exists $R_0 > 0$ such that $(1+|x|)^\gamma |g_n(x) - \bar g(x)| \le \frac{\varepsilon}{3}$ in $\mathbb{R}^N \setminus B_{R_0}$ uniformly in n. Using (3.23) instead we get that there exists r_0 such that $(1+|x|)^\gamma |g_n(x) - \bar g(x)| \le \frac{\varepsilon}{3}$ in B_{r_0} uniformly in n. Finally since, $v_n \to \bar v$ in $L^\infty(B_{R_0} \setminus B_{r_0})$ then we get that $(1+|x|)^\gamma |g_n - \bar g| < \frac{\varepsilon}{3}$ for n large enough in $B_{R_0} \setminus B_{r_0}$ and the proof of $i)$ is complete.

Let us prove $ii)$. Let us consider the linearized operator of $I - T$ in (λ, u_λ). We have that

$$< I - T_v'(\lambda, u_\lambda)I, w >= w - P_{K_\lambda}\left(\left(-\Delta - \frac{\lambda}{|x|^2}I\right)^{-1}\left(C(\lambda)\frac{N+2}{N-2}u_\lambda^{2^*-2}w\right)\right)$$

so that $w - T_v'(\lambda, u_\lambda)w = 0$ if and only if $w \in K_\lambda \cap X$ satisfies

$$-\Delta w - \frac{\lambda}{|x|^2}w - C(\lambda)\frac{N+2}{N-2}u_\lambda^{2^*-2}w = LC(\lambda)\frac{N+2}{N-2}u_\lambda^{2^*-2}Z_\lambda$$

weakly in $D^{1,2}(\mathbb{R}^N)$ for some $L = L(w) \in \mathbb{R}$. Multiplying by Z_λ and recalling the equation satisfied by Z_λ we get

$$0 = LC(\lambda)\frac{N+2}{N-2}\int_{\mathbb{R}^N} u_\lambda^{2^*-2} Z_\lambda^2$$

and this implies $L = 0$. Then $w \in K_\lambda$ is a weak solution of

$$-\Delta w - \frac{\lambda}{|x|^2}w - C(\lambda)\frac{N+2}{N-2}u_\lambda^{2^*-2}w = 0 \quad \text{in } \mathbb{R}^N. \tag{3.26}$$

Using Lemma 1.3 then we get that if $\lambda \neq \lambda_n$ equation (3.26) has only the solution Z_λ which is not in K_λ. This means that equation (3.26) has in K_λ only the solution $w = 0$ and the operator $I - T_v'(\lambda, u_\lambda)I$ is indeed invertible, concluding the proof. □

To prove the bifurcation result (Theorem 1.5) we need to exploit some of the symmetries of problem (1.16). So we define the subspace $\mathcal{H}$ of X given by

$$\mathcal{H} := \{v \in X\,, \text{s.t. } v(x_1,\dots,x_N) = v(g(x_1,\dots,x_{N-1}),x_N)\,,\ \text{for any } g \in O(N-1)\}.$$

Now let us consider the subgroups $\mathcal{G}_h$ of $O(N)$ defined by

$$\mathcal{G}_h = O(h) \times O(N-h) \quad \text{for } 1 \le h \le \left[\frac{N}{2}\right]$$

where $[a]$ stands for the integer part of a. We consider also the subspaces $\mathcal{H}^h$ of X of functions invariant by the action of $\mathcal{G}_h$.

The results of Smoller and Wasserman in [SW86] and [SW90] implies that, for any j the eigenspace of the Laplace Beltrami operator related to μ_j (see Section 3.2) contains only one eigenfunction which is $O(N-1)$-invariant (or which is invariant by the action of $\mathcal{G}_h$). Then, Corollary 1.4 implies that

$$m_{\mathcal{H}}(\lambda_j - \varepsilon) - m_{\mathcal{H}}(\lambda_j + \varepsilon) = 1$$

if ε is small enough, where $m_{\mathcal{H}}$ denotes the Morse index of u_λ in the space $\mathcal{H}$ (or $\mathcal{H}^h$).

The change of the Morse index of u_λ is a good clue to having the bifurcation, but since u_λ is radially degenerate we have to use the projection P_λ, changing problem (1.16) with problem (3.18).

What we can do, at this step, is to prove a bifurcation result for problem (3.18). To prove this we need "roughly speaking" that the Morse index of u_λ as a solution of problem (3.18) is the same as $m(\lambda)$ and this is proved in the following proposition.

Proposition 3.6. *The number of the eigenvalues of $T_v'(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ coincides with the morse index $m(\lambda)$ of u_λ.*

Proof. Λ is an eigenvalue for the linear operator $T_v'(\lambda, u_\lambda)I$ if and only if

$$\Lambda I - T_v'(\lambda, u_\lambda)I = 0$$

has a nontrivial solution in $X \cap K_\lambda$. This means that we have to find $w \in X \cap K_\lambda$, $w \neq 0$ which verifies

$$-\Delta w - \frac{\lambda}{|x|^2} w = \frac{1}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} w + \frac{L}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} Z_\lambda \quad \text{in } \mathbb{R}^N \tag{3.27}$$

for some $L = L(w) \in \mathbb{R}$ and $\frac{1}{\Lambda} \in (0,1)$.
Observe that, since $\Lambda \neq 1$, the function $w_1 = \frac{L}{\Lambda - 1} Z_\lambda$ is always a solution of (3.27) (that does not belong to K_λ) and all the other solutions of (3.27) are given by $w = w_1 + \tilde{w}$ where $\tilde{w} \in X \cap K_\lambda$ satisfies

$$-\Delta \tilde{w} - \frac{\lambda}{|x|^2} \tilde{w} = \frac{1}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} \tilde{w} \quad \text{in } \mathbb{R}^N. \tag{3.28}$$

Now, if $\frac{1}{\Lambda}$ is not an eigenvalue of (3.28) then $\tilde{w} = 0$ and (3.27) has only the solution w_1. But w_1 is not in K_λ so that $w = 0$ and $L = 0$ in (3.27).
If else, $\frac{1}{\Lambda}$ is an eigenvalue of (3.28) and $\tilde{w}$ a corresponding eigenfunction we can use $\tilde{w}$ as a test function in (3.27), Z_λ as a test function in (3.28), getting that

$$\int_{\mathbb{R}^N} u_\lambda^{2^*-2} Z_\lambda \tilde{w} \, dx = 0$$

so that $\tilde{w} \in K_\lambda$. Since $w_1 \notin K_\lambda$ this implies $L = 0$ in (3.27) so that equation (3.27) coincides with equation (3.28).
We have shown so far that the number of the eigenvalues of $T'_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ is equal to the number of the eigenvalues of (3.28) counted with multiplicity in $(0,1)$, and this is the Morse index of u_λ. □

From Proposition 3.6 we have that the number of the eigenvalues of $T'_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ decreases by one going from $\lambda_j - \varepsilon$ to $\lambda_j + \varepsilon$ and ε small enough in the space $\mathcal{H}$ (or $\mathcal{H}^h$) and this is sufficient to have the bifurcation.
We do not give the details of the global bifurcation result for problem (3.18), we only sketch the proof of the local bifurcation result to have an idea how to use the results of Propositions (3.5) and (3.6).
Then the global bifurcation result will follow reasoning as in [G, Theorem 3.3], (see also [AM]).

Proposition 3.7. *The points $(\lambda_j, u_{\lambda_j})$ are nonradial bifurcation points for the curve (λ, u_λ) of radial solutions of (3.18).*

Proof. Assume by contradiction that $(\lambda_j, u_{\lambda_j})$ is not a bifurcation point for (3.18), for some j. Then there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ and $\forall c \in (0, \varepsilon_0)$

$$I - T(\lambda, v) \neq 0$$

for any $\lambda \in (\lambda_j - \varepsilon, \lambda_j + \varepsilon) \subset (-\infty, 0)$ and for any $v \in \mathcal{H}$ (or in $\mathcal{H}^h$) such that $\|v - u_\lambda\|_X \leq c$ and $v \neq u_\lambda$.

Let $\Gamma := \{(\lambda, v) \in (\lambda_j - \varepsilon, \lambda_j + \varepsilon) \times \mathcal{H} : \|v - u_\lambda\|_X \le c\}$ and $\Gamma_\lambda := \{v \in \mathcal{H} \text{ s.t. } (\lambda, v) \in \Gamma\}$. By the homotopy invariance of the Leray Schauder degree we have that

$$deg\,(I - T(\lambda, \cdot), \Gamma_\lambda, 0) \text{ is constant on } (\lambda_j - \varepsilon, \lambda_j + \varepsilon). \tag{3.29}$$

Since the linearized operator is invertible for $\lambda = \lambda_j - \varepsilon$ and $\lambda = \lambda_j + \varepsilon$ we can compute the Leray Schader degree and we have that

$$deg\left(I - T(\lambda_j \pm \varepsilon, \cdot), \Gamma_{\lambda_j \pm \varepsilon}, 0\right) = (-1)^{\beta(\lambda_j \pm \varepsilon)}$$

where $\beta(\lambda)$ is the number of the eigenvalues of $T'_v(\lambda, u_\lambda)$ counted with multiplicity contained in $(1, +\infty)$, see [AM, Theorem 3.20]. Then Proposition 3.6 implies that

$$deg\left(I - T(\lambda_j - \varepsilon, \cdot), \Gamma_{\lambda_j - \varepsilon}, 0\right) = -deg\left(I - T(\lambda_j + \varepsilon, \cdot), \Gamma_{\lambda_j + \varepsilon}, 0\right)$$

contradicting (3.29). Then $(\lambda_j, u_{\lambda_j})$ is a bifurcation point for (3.18) and the bifurcating solutions are nonradial since u_λ is radially nondegenerate in K_λ. □

Finally we can state the global bifurcation result for (3.18).

Proposition 3.8. *Let us fix $j \in \mathbb{N}$ and let λ_j be as defined in (1.22). Then*
i) If j is odd there exists at least a continuum of nonradial solutions to (3.18), invariant with respect to $O(N-1)$, bifurcating from the pair $(\lambda_j, u_{\lambda_j})$.
ii) If j is even there exist at least $\left[\frac{N}{2}\right]$ continua of nonradial solutions to (3.18) bifurcating from the pair $(\lambda_j, u_{\lambda_j})$. The first branch is $O(N-1)$ invariant, the second is $O(N-2) \times O(2)$ invariant, etc.

The final step for the proof of Theorem 1.5 will be to show that the solutions we have found in Theorem 3.8 are indeed solutions of (1.16). This will be done in the next section.

3.4. **The Lagrange multiplier is zero.** In the previous section we proved the existence of solutions $u_{\varepsilon,n}$ and parameters $\lambda_{\varepsilon,n}, L_\varepsilon$ verifying

$$-\Delta u_{\varepsilon,n} - \frac{\lambda_{\varepsilon,n}}{|x|^2} u_{\varepsilon,n} - C(\lambda_{\varepsilon,n}) u_{\varepsilon,n}^{2^*-1} = L_\varepsilon C(\lambda_{\varepsilon,n}) \frac{N+2}{N-2} u_{\varepsilon,n}^{2^*-2} Z_{\varepsilon,n} \text{ in } \mathbb{R}^N \tag{3.30}$$

where $Z_{\varepsilon,n} = Z_{\lambda_{\varepsilon,n}}$, with $\lambda_{\varepsilon,n}$ and $u_{\varepsilon,n}$ and L_ε such that $\lambda_{0,n} = \lambda_n$ and $u_{\varepsilon,n} = u_{\lambda_n}$ and $L_0 = 0$.
In the following we denote by C a generic constant (independent of n and ε) which can change from line to line.
First we prove a bound on L_ε.

Lemma 3.9. *We have*

$$|L_\varepsilon| \le C.$$

Proof. Using $Z_{\varepsilon,n}$ as a test function in (3.30) we get

$$L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}\int\limits_{\mathbb{R}^N} u_{\lambda_{\varepsilon,n}}^{2^*-2} Z_{\varepsilon,n}^2 = \int\limits_{\mathbb{R}^N} \nabla u_{\varepsilon,n}\cdot\nabla Z_{\varepsilon,n}$$
$$-\int\limits_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}Z_{\varepsilon,n} - C(\lambda_{\varepsilon,n})\int\limits_{\mathbb{R}^N} u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n}.$$

Using Lemma 5.1, the Holder and the Sobolev inequality we get

$$L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}\int\limits_{\mathbb{R}^N} u_{\lambda_{\varepsilon,n}}^{2^*-2} Z_{\varepsilon,n}^2 \le C\|u_{\varepsilon,n}\|_{1,2}\|Z_{\varepsilon,n}\|_{1,2}$$

so that the claim follows. □

Proposition 3.10. *Let $u_{\varepsilon,n}$ be the solution of (3.30). Then $L_\varepsilon = 0$ in (3.30) for ε small enough.*

Proof. Applying the Pohozaev identity (5.4) with $f(x,u) = \frac{\lambda_{\varepsilon,n}}{|x|^2}u + C(\lambda_{\varepsilon,n})u^{2^*-1} + L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}u^{2^*-2}Z_{\varepsilon,n}$ we get

$$\int\limits_{\mathbb{R}^N}|\nabla u_{\varepsilon,n}|^2 - \frac{N}{N-2}\int\limits_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}^2 - C(\lambda_{\varepsilon,n})\int\limits_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*}$$
$$-\frac{2N}{N-2}L_\varepsilon C(\lambda_{\varepsilon,n})\int\limits_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n} + \frac{2}{N-2}\int\limits_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}^2$$
$$-\frac{2}{N-2}L_\varepsilon C(\lambda_{\varepsilon,n})\int\limits_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*-1}x\cdot\nabla Z_{\varepsilon,n} = 0.$$

Using $u_{\varepsilon,n}$ as a test function in (3.30) then we get

$$L_\varepsilon\int\limits_{\mathbb{R}^N}\left(u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n} + u_{\varepsilon,n}^{2^*-1}x\cdot\nabla Z_{\varepsilon,n}\right) = 0$$

and this implies $L_\varepsilon = 0$ if we show that the integral is different from zero. Let us recall that $u_{\varepsilon,n}\to u_{\lambda_n}$, $Z_{\varepsilon,n}\to Z_{\lambda_n}$ as $\varepsilon\to 0$. Moreover, by the definition of $Z_{\varepsilon,n}$ and since $\lambda_n < 0$ we get $Z_{\varepsilon,n} = O\left((1+|x|)^{2-N}\right)$ and $|\nabla Z_{\varepsilon,n}| = O\left((1+|x|)^{1-N}\right)$. Finally since $u_{\varepsilon,n}\in X$ we have that $u_{\varepsilon,n}\in X$ and then $u_{\varepsilon,n} = O\left((1+|x|)^{\gamma}\right)$ with $\frac{N-2}{2} < \gamma < N-2$. So by the dominate convergence theorem we derive that

$$\int\limits_{\mathbb{R}^N}\left(u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n} + u_{\varepsilon,n}^{2^*-1}x\cdot\nabla Z_{\varepsilon,n}\right)\to\int\limits_{\mathbb{R}^N}u_{\lambda_n}^{2^*-1}x\cdot\nabla Z_{\lambda_n}\neq 0$$

so that $L_\varepsilon = 0$ if ε small enough, concluding the proof. □

4. The subcritical case $1 < p < \frac{N+2}{N-2}$

Let us start this section recalling some known facts. Next theorem collects some results of different authors (see [GNN] and [S]).

Theorem 4.1. *Let $1 < p < \frac{L+2}{L-2}$ and v_L be the unique positive solution of*

$$\begin{cases} -v'' - \frac{L-1}{r}v' = v^p & \text{in } (0,1) \\ v > 0 & \text{in } (0,1) \\ v'(0) = v(1) = 0 \end{cases} \tag{4.1}$$

where L is a real number greater than 1. Then v_L is nondegenerate and its Morse index is 1.

Remark 4.2. Theorem 4.1 allows to establish the existence of the branch of radial solutions u_λ as stated in Theorem 1.8 in the Introduction. Moreover, using the transformation $\mathcal{L}_p$ we are able to find the behaviour of the radial solution u_λ at zero, see (1.30).

Remark 4.3. As $\lambda > 0$ problem (1.28) has been studied in [CG]. In this case the authors proved the existence of a unique radial solution u_λ and his behaviour near the origin, which is exactly the same as in (1.30). Their proof relies on the the moving plane method, which ensures that every positive solution is radial, and on the phase plane analysis of the radial solutions. Both steps strongly rely on the hypothesis that $\lambda > 0$ and cannot be extended to $\lambda \leq 0$. Using the map $\mathcal{L}_p$ we easily obtain a new proof of the results of [CG] and we extend them to the case $\lambda < 0$.

Remark 4.4. The nondegeneracy result in Theorem 4.1 and the implicit function theorem imply that the function $\lambda \to v_\lambda$ is C^1.

Proof of Theorem 1.8. Let $u(r)$ be a radial solution to (1.28) and let $v(r) = \mathcal{L}_p(u(r))$ as defined in (1.2). From Proposition 1.1 we know that the transformed function $v(r)$ satisfies

$$\begin{cases} -v'' - \frac{M-1}{r}v' = A(\lambda,p)v^p & \text{in } (0,1) \\ v > 0 & \text{in } (0,1) \\ v(1) = 0 \end{cases} \tag{4.2}$$

with M as in (1.8) and $A(\lambda,p)$ as in (1.9).

Moreover, a straightforword computation shows that if $1 < p < \frac{N+2}{N-2}$ then $1 < p < \frac{M+2}{M-2}$. Then, by (1.13) we have that

$$\int_0^1 r^{M-1}\left(v'(r)\right)^2 = \frac{1}{\nu_\lambda}\int_0^1 s^{N-1}\left(u'(s)^2 - \frac{\lambda}{s^2}u^2(s)\right) \leq C. \tag{4.3}$$

We want to use (4.3) to prove that the function v satisfies $v'(0) = 0$ also. This will imply the existence and uniqueness result using Theorem 4.1 with

$L = M$.
To this end we let $\widetilde{v}(r) = \frac{1}{r^{M-2}} v\left(\frac{1}{r}\right)$. The function $\widetilde{v}$ solves the equation

$$\begin{cases} -(r^{M-1}\widetilde{v}'(r))' = A(\lambda, p) r^{(M-2)p-3} \widetilde{v}^p(r) & \text{in } (1, +\infty) \\ \widetilde{v}(1) = 0 \end{cases} \tag{4.4}$$

and satisfies

$$\int_1^{+\infty} r^{M-1} \left(\widetilde{v}'(r)\right)^2 \leq C.$$

Reasoning exactly as in the radial Lemma of Ni (see [Ni]) then we get that

$$\widetilde{v}(r) \leq C r^{\frac{2-M}{2}} \tag{4.5}$$

so that $\widetilde{v}(r) \to 0$ as $r \to +\infty$ since $M > 2$. Let r_0 be a maximum point for $\widetilde{v}$ in $(1, +\infty)$. Integrating (4.4) in (r_0, r) then we get

$$-r^{M-1}\widetilde{v}'(r) = A(\lambda, p) \int_{r_0}^{r} s^{(M-2)p-3} \widetilde{v}^p(s)\, ds.$$

Using estimate (4.5) we have

$$\Big| \int_{r_0}^{r} s^{(M-2)p-3} \widetilde{v}^p(s)\, ds \Big| \leq C \Big| \int_{r_0}^{r} s^{\frac{M-2}{2}p-3}\, ds \Big|. \tag{4.6}$$

This implies that

$$|\widetilde{v}'(r)| \leq \begin{cases} C r^{1-M} & \text{when } p < \frac{4}{M-2} \\ C r^{1-M} \log r & \text{when } p = \frac{4}{M-2} \\ C r^{-1-M+\frac{M-2}{2}p} & \text{when } p > \frac{4}{M-2} \end{cases} \tag{4.7}$$

When $p < \frac{4}{M-2}$ (4.7) produces the optimal decay for $\widetilde{v}'(r)$. Otherwise, if $p \geq \frac{4}{M-2}$ we need to repeat the procedure estimating again the integral in (4.6) using (4.7). In any case after a finite number of steps we get that

$$|\widetilde{v}'(r)| \leq C r^{1-M} \tag{4.8}$$

and this implies that

$$\widetilde{v}(r) \leq C r^{2-M}. \tag{4.9}$$

Turning back to the function v then we get that

$$v(r) \leq C \text{ in } [0, 1].$$

Further, using the definition of $\widetilde{v}$ and estimates (4.8) and (4.9) we have

$$\lim_{r \to 0} r^{M-1} v'(r) = \lim_{r \to 0} -\frac{1}{r} \widetilde{v}'\left(\frac{1}{r}\right) + (2 - M)\widetilde{v}(\frac{1}{r}) = 0$$

since $M>2$. Integrating equation (4.2) then we obtain that

$$-r^{M-1}v'(r)=A(\lambda,p)\int_0^r s^{M-1}v^p(s)\,ds \tag{4.10}$$

so that $v'(r)<0$ in $(0,1)$ and $\lim_{r\to 0}v(r)$ exists and it is finite showing that v is continuous at the origin. Using (4.10) again we have

$$\lim_{r\to 0}v'(r)=-A(\lambda,p)\lim_{r\to 0}\frac{\int_0^r s^{M-1}v^p(s)\,ds}{r^{M-1}}=-A(\lambda,p)\lim_{r\to 0}\frac{r}{M-1}v^p(r)=0.$$

This shows that the transformed function $v(r)$ has to be a solution of (4.1) since the constant $A(\lambda,p)$ can be merged into the equation. Theorem 4.1 then implies the existence and uniqueness of the radial solution. The final estimate follows inverting the transformation $\mathcal{L}_p$ and using the continuity of $v(r)$ in 0. □

Corollary 4.5. *We have that the radial solution u_λ to (1.28) satisfies $u_\lambda(0)=0$ if $\lambda<0$.*

Proof. It is enough to remark that $\nu_\lambda>1$ as $\lambda<0$. Then the claim follows by (1.30). □

In the rest of the section we will denote by u_λ the unique radial solution to (1.28) and by

$$H=\left\{u\in H^1((0,1),r^{M-1}dr)\text{ such that }u(1)=0\right\}.$$

Set $v_\lambda(r)$ as in (1.2) and $\Lambda(\lambda)$ be as defined in (1.31). Although the embedding of $H\hookrightarrow L^2((0,1),r^{M-3}dr)$ is not compact, $\Lambda(\lambda)$ is achieved. This is a consequence of Proposition 5.8 in the Appendix, whose proof is basically the same of Proposition A.1 in [GGN]. Then we have the following result:

Corollary 4.6. *The first eigenvalue $\Lambda(\lambda)$ defined in (1.31) is achieved.*

Proof. Since $\Lambda(\lambda)\le(1-p)\frac{\int_0^1 v_\lambda^{p+1}r^{M-1}}{\int_0^1 v_\lambda^2 r^{M-3}}<0$ the claim follows by Proposition 5.8. □

As in the previous section we study the linearized operator at the solution u_λ and we recall that u_λ is non degenerate if the linear problem

$$\begin{cases}-\Delta w-\frac{\lambda}{|x|^2}w=pu_\lambda^{p-1}w & \text{in } B_1\\ w\in H_0^1(B_1)\end{cases} \tag{4.11}$$

admits only the trivial solution.

Theorem 4.7. *Let $k\in\mathbb{N}$, $k\ge 1$ and $\lambda\le\frac{(N-2)^2}{4}$. The linearized equation at the radial solution u_λ, i.e. equation (4.11), admits a solution if and only if λ satisfies*

$$-\Lambda(\lambda)=\frac{16k(N-2+k)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2}, \tag{4.12}$$

for some $k \geq 1$. Moreover the space of solutions of (4.11) corresponding to a value of λ which satisfies (4.12) related to some k, has dimension $\frac{(N+2k-2)(N+k-3)!}{(N-2)!\,k!}$ and it is spanned by

$$Z_{k,i,\lambda}(x) = \frac{1}{|x|^{\frac{a}{b}}}\psi_1\left(|x|^{\frac{1}{b}}\right)Y_{k,i}(x)$$

where ψ_1 is the positive eigenfunction associated to $\Lambda(\lambda)$ and $\{Y_{k,i}\}$, $i = 1, \dots, \frac{(N+2k-2)(N+k-3)!}{(N-2)!k!}$, form a basis of $\mathbb{Y}_k(\mathbb{R}^N)$, the space of all homogeneous harmonic polynomials of degree k in $\mathbb{R}^N$.
Finally, for every $k \geq 1$ there exists at least one value of λ that satisfies (4.12) and if λ is not a solution to (4.12) then the solution u_λ is nondegenerate.

Proof. The beginning of the proof is basically the same as Lemma 1.3. Let v be a solution to (4.11) and decomposing as sum of spherical harmonics we reduce ourselves to study the following ODE,

$$\begin{cases} -\psi_k''(r) - \frac{N-1}{r}\psi_k'(r) + \frac{\mu_k-\lambda}{r^2}\psi_k(r) = pu_\lambda^{p-1}(r)\psi_k(r) & \text{in } (0,1) \\ \psi_k(1) = 0\ ,\ \int_0^1 r^{N-1}\left(\psi_k'(r)\right)^2 dr < \infty \end{cases}$$

where $\mu_k = k(N-2+k)$. Setting again, $\widehat{\psi}_k(r) = r^a\psi_k\left(r^b\right)$ we have that $\widehat{\psi}_k$ solves

$$\begin{cases} -\widehat{\psi}_k''(r) - \frac{M-1}{r}\widehat{\psi}_k'(r) + \frac{b^2\mu_k}{r^2}\widehat{\psi}_k(r) = pv_\lambda^{p-1}(r)\widehat{\psi}_k & \text{in } (0,1) \\ \widehat{\psi}_k(1) = 0\ ,\ \widehat{\psi}_k \in H \end{cases} \tag{4.13}$$

Note that, since v_λ is nondegenerate, from Theorem 4.1, the previous problem cannot have solutions for $k = 0$. So we assume that $k \geq 1$.
By Theorem 4.1 we get that (4.13) has a nontrivial solution belonging to the space H if and only if $-b^2\mu_k = \Lambda(\lambda)$ which is the unique negative eigenvalue. Moreover by Lemma 5.9 we get that $\widehat{\psi}_k \in L^\infty(0,1)$. Recalling (1.4) we get that equation (4.11) admits a solution if and only if

$$-\Lambda(\lambda) = \frac{16k(N-2+k)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2} \tag{4.14}$$

for some $k \geq 1$. Since the solution u_λ is not explicitly known as in the previous section, we have to show that (4.14) has at least a solutions. Let us consider the two *limit* cases $\lambda = 0$ and $\lambda = -\infty$. Note that by Remark 4.4 we derive that $\Lambda(\lambda)$ is a continuous function of λ.
Case i) $\lambda = 0$.

First let us study the limit of the solution v_λ to (1.29) as λ goes to zero. By the uniqueness result of Theorem 4.1 we have that v_λ can be characterized as

$$\inf_{\int_0^1 v(r)^{p+1}r^{M-1}=1}\int_0^1\left(v'(r)\right)^2 r^{M-1} \quad v \in H.$$

Then it is easy to see that v_λ achieves this infimum and then $\int_0^1 \left(v_\lambda'(r)\right)^2 r^{M-1} \le C$ for some positive constant C independent of λ. So $v_\lambda \rightharpoonup v_0$ where, from Remark 4.4, v_0 satisfies

$$\begin{cases} -v'' - \frac{N-1}{r} v' = v^p & \text{in } (0,1) \\ v > 0 & \text{in } (0,1) \\ v'(0) = v(1) = 0, \end{cases}$$

since $A(0,p) = 1$. Then we get, for any $k \ge 1$,

$$\lim_{\lambda \to 0} \left(\Lambda(\lambda) + \frac{16k(N-2+k)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2} \right) =$$
$$\inf_{v \in H} \frac{\int_0^1 \left(|v'|^2 - p v_0^{p-1} v^2\right) r^{N-1}}{\int_0^1 v^2 r^{N-3}} + k(N-2+k) > 0$$

because $\inf\limits_{v \in H} \frac{\int_0^1 \left(|v'|^2 - p v_0^{p-1} v^2\right) r^{N-1}}{\int_0^1 v^2 r^{N-3}} > 1 - N$, comparing the eigenfunction which achieves $\Lambda(0)$ with v_0' and using the maximum principle.
Case ii) $\lambda = -\infty$.

For any $k \ge 1$, testing $\Lambda(\lambda)$ with v_λ we get

$$\begin{aligned} &\Lambda(\lambda) + \frac{16k(N-2+k)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2} \le \\ &(1-p)A(\lambda,p)\frac{\int_0^1 v_\lambda^{p+1} r^{M-1}}{\int_0^1 v_\lambda^2 r^{M-3}} + o(1) \end{aligned} \tag{4.15}$$

for λ large enough. By (4.2) we have that

$$\int_0^1 |v_\lambda'|^2 r^{M-1} = A(\lambda,p) \int_0^1 v_\lambda^{p+1} r^{M-1}$$

and using the Hardy inequality for radial function (see [GP]),

$$\int_0^1 v^2 r^{M-3} \le \left(\frac{M-2}{2}\right)^2 \int_0^1 |v'|^2 r^{M-1}$$

we get that (4.15) becomes

$$\Lambda(\lambda) + \frac{16k(N-2+k)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2} \le$$
$$\frac{1-p}{\left(\frac{M-2}{2}\right)^2} + o(1) = \frac{1-p}{\left(\frac{2}{p-1}\right)^2} + o(1) < 0 \quad \text{for } \lambda \text{ large enough.}$$

By Cases i) and ii) we derive that, for any $k \geq 1$, there exists at least one value of λ which solves (4.14). $\square$

Corollary 4.8. *The Morse index $m(\lambda)$ of u_λ is equal to*

$$m(\lambda) = \sum_{\substack{0 \leq j < \frac{2-N}{2} + \frac{1}{2}\sqrt{(N-2)^2 - 4\frac{\Lambda(\lambda)}{b^2}} \\ j \ integer}} \frac{(N+2j-2)(N+j-3)!}{(N-2)!\, j!}.$$

In particular, we have that $m(\lambda) \to +\infty$ as $\lambda \to -\infty$.

Proof. Reasoning exactly as in the proof of Proposition 1.4 we consider the eigenvalue problem with weight and we call Γ_i the corresponding eigenvalues. Then, we have that the linearized equation has a negative eigenvalue with weight Γ_i if and only if

$$\Lambda(\lambda) = b^2 \left(\Gamma_i - \mu_j\right)$$

for some $j \in \mathbb{N}$. So we have that the indexes j which contribute to the Morse index of the solution u_λ are those that satisfies

$$\Gamma_i = \frac{\Lambda(\lambda)}{b^2} + \mu_j < 0 \tag{4.16}$$

for some $j \in \mathbb{N}$. This implies, recalling the value of μ_j that $j < \frac{2-N}{2} + \frac{1}{2}\sqrt{(N-2)^2 - 4\frac{\Lambda(\lambda)}{b^2}}$. The claim follows recalling the dimension of the eigenspace of the Laplace-Beltrami operator related to μ_j. The last assertion follows since $\Lambda(\lambda) \to -\infty$ for $\lambda \to -\infty$. $\square$

From Theorem 4.7 and Corollary 4.8 we have that if λ^* satisfies (4.12) and the function $\Lambda(\lambda) + b^2\mu_k$ changes sign at the endpoints of a suitable interval containing λ^*, then the Morse index of the radial solution u_λ changes. This change in the Morse index is responsible of the bifurcation. From the continuity of $\Lambda(\lambda)$ we know that there should exists at least one value λ_k that satisfies (4.12) for every $k \geq 1$ but since we do not know if the function $\Lambda(\lambda)$ is analytic we cannot say that these values λ_k are isolated. To overcame this problem, in the next Proposition we construct an interval $I_k = [\alpha_k, \beta_k]$ which contains at least one of the points λ_k that satisfies (4.12) and at which the function $\Lambda(\lambda) + b^2\mu_k$ changes sign, and such that the Morse index of the radial solution u_λ at the value α_k and β_k differs from $\frac{(N+2k-2)(N+k-3)!}{(N-2)!\,k!}$ which is the dimension of the eigenspace of the Laplace Beltrami operator related to the eigenvalue μ_k.

Proposition 4.9. *There exists a sequence λ_k verifying*

$$-\Lambda(\lambda_k) = \frac{16k(N+k-2)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda_k}\right)+4\right]^2}, \tag{4.17}$$

and a sequence of intervals $I_k=[\alpha_k,\beta_k]\subset(-\infty,0)$ with $\lambda_k\in I_k$ such that

$$\Lambda(\beta_k) > -\frac{16k(N+k-2)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4(\beta_k)}\right)+4\right]^2}, \tag{4.18}$$

$$\Lambda(\alpha_k) < -\frac{16k(N+k-2)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4(\alpha_k}\right)+4\right]^2}, \tag{4.19}$$

and

$$\Lambda(\beta_k) < -\frac{16h(N-2+h)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\beta_k}\right)+4\right]^2}, \tag{4.20}$$

for any $h<k$ while

$$\Lambda(\alpha_k) > -\frac{16j(N-2+j)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\alpha_k}\right)+4\right]^2}, \tag{4.21}$$

for any $j>k$.

Proof. In order to simplify the notation we consider first the case $k=1$. Set, for $\lambda\le 0$,

$$L(\lambda)=\Lambda(\lambda)\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2$$

and define λ_1 as

$$\lambda_1=\sup_{\lambda\le 0} I_{1,\lambda}$$

where

$$I_{1,\lambda}=\{\lambda\le 0 \text{ such that } L(\lambda)=-16(N-1)\}.$$

By cases i) and ii) in Theorem 4.7 we get that $I_{1,\lambda}\ne\emptyset$ and since L is a continuous function we have that there exists λ_1 such that

$$L(\lambda_1)=-16(N-1),$$

and any other point $\lambda^*\ne\lambda_1$ which satisfies

$$L(\lambda^*)=-16(N-1),$$

must verify

$$\lambda^*<\lambda_1.$$

Analogously we define, for $k\ge 2$

$$\lambda_k=\sup_{\lambda\le 0} I_{k,\lambda} \tag{4.22}$$

where

$$I_{k,\lambda}=\{\lambda\le 0 \text{ such that } L(\lambda)=-16k(N-2+k)\}.$$

As in the previous case, using the proof of Theorem 4.7 we get that there exists λ_k such that

$$L(\lambda_k)=-16k(N-2+k),$$

and λ_k achieves (4.22).

Let us show that

$$\lambda_{k+1} < \lambda_k \quad \text{for any } k \geq 1.$$

Since the function $16k(N-2+k)$ is strictly increasing in k we cannot have that $\lambda_{k+1} = \lambda_k$. So by contradiction let us suppose that

$$\lambda_{k+1} > \lambda_k,$$

for some $k \geq 1$. Then,

$$L(\lambda_{k+1}) = -16(k+1)(N-1+k) < -16k(N-2+k) \tag{4.23}$$

and by case i) of the proof of Theorem 4.7 we have

$$\lim_{\lambda\to 0} L(\lambda) = \lim_{\lambda \to 0} \Lambda(\lambda)\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4\lambda}\right)+4\right]^2 = \\ (16+o(1))\lim_{\lambda\to 0}\Lambda(\lambda) > -16(N-1) \geq -16k(N-2+k)$$

by the intermediate value Theorem for continuous function we get that there exists $\tilde{\lambda}_k \geq \lambda_{k+1}$ such that

$$L(\tilde{\lambda}_k) = -16k(N-2+k)$$

and this is a contradiction with the definition of λ_k.

So we have shown that

$$0 > \lambda_1 > \lambda_2 > .. > \lambda_k > ..$$

Now we prove the claim: again by case i) of Theorem 4.6, since

$$\lim_{\lambda\to 0} L(\lambda) > -16(N-1)$$

we get that there exists $\lambda_1 < \beta_1 < 0$ such that

$$L(\beta_1) > -16(N-1)$$

and this implies $\Lambda(\beta_1) > -\frac{16(N-1)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4(\beta_1)}\right)+4\right]^2}$. This proves (4.18).

On the other hand, since by (4.23) we have that $L(\lambda_2) = -32N < -16(N-1) = L(\lambda_1)$ then there exists $\lambda_2 < \alpha_1 < \lambda_1$ such that $L(\alpha_1) < -16(N-1)$, which implies $\Lambda(\alpha_1) < -\frac{16(N-1)}{\left[(p-1)\left(2-N+\sqrt{(N-2)^2-4(\alpha_1)}\right)+4\right]^2}$. This proves (4.19).

Finally since $\sup_{k\geq 2} \lambda_k = \lambda_2 < \alpha_1$ we have that $L(\alpha_1) > -32N \geq -16j(N-2+j)$ for any $j > 1$ so that (4.21) follows.

Now we explain how to pass from $k=1$ to $k=2$. We take $\beta_2 = \alpha_1 \in (\lambda_2, \lambda_1)$. Then from (4.19) and (4.21) we have that

$$L(\beta_2) = L(\alpha_1) > -32N$$

and

$$L(\beta_2) = L(\alpha_1) < -16(N-2)$$

so that (4.18) and (4.20) follows for $k=2$.
From (4.23) we have that $L(\lambda_3) = -48(N+1) < -32N = L(\lambda_2)$. Then there exists $\lambda_3 < \alpha_2 < \lambda_2$ such that $-48(N+1) < L(\alpha_2) < -32N$ so that $L(\alpha_2) < -32N$ and this proves (4.19) for $k=2$. Finally by the choice of α_2 we have $L(\alpha_2) > -48(N+1) \geq -16j(N-2+j)$ for any $j>2$ so that (4.21) is proved for $k=2$. The general case can be carried out with the same proof. □

As in Section 3.3 one can define the operator $T(\lambda, v) : (-\infty, 0) \times H_0^1(B_1) \cap L^\infty(B_1) \longrightarrow H_0^1(B_1) \cap L^\infty(B_1)$ as $T(\lambda, v) = \left(-\Delta - \frac{\lambda}{|x|^2} I\right)^{-1} ((v^+)^p)$ and look for zeros of $I - T(\lambda, v)$. Letting $X = H_0^1(B_1) \cap L^\infty(B_1)$ and reasoning as in the proof of Lemma 3.4, we have that the operator T is well defined from $(-\infty, 0) \times X$ into X. T is continous with respect to λ and it is compact from X into X for any $\lambda \in (-\infty, 0)$ fixed.
Moreover the linearized operator $I - T'(\lambda, u_\lambda) I$ is invertible for any value of λ which do not satisfy (4.12).
To prove the bifurcation we have to consider as in the previous section the suspace $\mathcal{H}$ of X of functions which are $O(N-1)$-invariant and the subspaces $\mathcal{H}^h$ of X of functions which are invariant by the action of $\mathcal{G}_h$.
Using these spaces by Theorem 4.7 and Proposition (4.9) we deduce the following result

Proposition 4.10. *For every $k \in \mathbb{N}$ the curve of radial solution $(\lambda, u_\lambda) \in (-\infty, 0) \times X$ contains a nonradial bifurcation point in the interval $I_k \times \mathcal{H}$, where I_k is as defined in Proposition (4.9).*
Moreover if k is even, for every $h = 1, \dots, \left[\frac{N}{2}\right]$ there exists a continuum of nonradial solution bifurcating from (λ, u_λ) in the interval $I_k \times \mathcal{H}^h$.

Proof. The proof is by contradiction. We consider only the case of the space $\mathcal{H}$. The other case follows in a very similar way.
Assume by contradiction that the curve (λ, u_λ) does not contain any bifurcation point in the interval $I_k \times \mathcal{H}$. Then there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and every $c \in (0, \varepsilon_0)$ we have

$$v - T(\lambda, v) \neq 0, \quad \forall \lambda \in I_k, \forall v \in X \text{ such that } 0 < \|v - u_\lambda\|_X \leq c. \tag{4.24}$$

Let us consider the set $\mathcal{C} := \{(\lambda, v) \in I_k \times X \ : \ \|v - u_\lambda\|_X < c\}$ and $\mathcal{C}_\lambda := \{v \in X \text{ such that } (\lambda, v) \in \mathcal{C}\}$. From (4.24) it follows that there exist no solutions of $v - T(\lambda, v) = 0$ on $\partial_{I_k \times X} \mathcal{C}$ different from the radial ones. By the homotopy invariance of the degree, we get

$$deg\,(I - T(\lambda, \cdot), \mathcal{C}_\lambda, 0) \text{ is constant on } I_k. \tag{4.25}$$

Moreover from (4.12), (4.18), (4.19), (4.20), and (4.21) we have that the linearized operator $T'(\lambda, u_\lambda)$ is invertible for $\lambda = \alpha_k$ and $\lambda = \beta_k$. Then

$$deg\,(I - T(\beta_k, \cdot), \mathcal{C}_{\beta_k}, 0) = (-1)^{m_{\mathcal{H}}(\beta_k)}$$

and

$$deg\,(I - T(\alpha_k, \cdot), \mathcal{C}_{\alpha_k}, 0) = (-1)^{m_{\mathcal{H}}(\alpha_k)}$$

where $m_{\mathcal{H}}(\lambda)$ denotes the Morse index of the radial solution u_λ in the space $\mathcal{H}$. By the choice of the space $\mathcal{H}$ we know that the eigenspace of the Laplace Beltrami operator associated to μ_k is one-dimensional. Then, repeating the proof of Corollary 4.8 in the space $\mathcal{H}$ we have that

$$m_{\mathcal{H}}(\lambda) = \begin{cases} 1 + \sup\{j \in \mathbb{N} \text{ s.t. } \Lambda(\lambda) + b^2\mu_j < 0\} & \text{if } \lambda \text{ does not satisfy (4.12)} \\ \\ \sup\{j \in \mathbb{N} \text{ s.t. } \Lambda(\lambda) + b^2\mu_j < 0\} & \text{if } \lambda \text{ satisfies (4.12)} \end{cases}$$

Then, from (4.18)-(4.21) we have that $m_{\mathcal{H}}(\beta_k) = 1 + (k-1) = k$ and $m_{\mathcal{H}}(\alpha_k) = 1 + k$ so that

$$deg\,(I - T(\beta_k, \cdot), \mathcal{C}_{\beta_k}, 0) = -deg\,(I - T(\alpha_k, \cdot), \mathcal{C}_{\alpha_k}, 0)$$

contradicting (4.25). Then, in the interval $I_k \times X$ there exists a bifurcation point for the curve (λ, u_λ) and the bifurcating solutions are nonradial since u_λ is radially nondegenerate. $\square$

5. Appendix

Lemma 5.1. *Let $\lambda \in (-\infty, \frac{(N-2)^2}{4})$. Then*

$$\left(\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} v^2\,dx \right)^{\frac{1}{2}} \tag{5.1}$$

is a norm on $D^{1,2}(\mathbb{R}^N)$ which is equivalent to the standard one.

Proof. It follows by the Hardy inequality distinguishing the two different cases, $\lambda > 0$ and $\lambda \le 0$. $\square$

Lemma 5.2. *Let $f(x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and let $\lambda \in (-\infty, \frac{(N-2)^2}{4})$. Then the equation*

$$-\Delta v - \frac{\lambda}{|x|^2} v = f \qquad \text{in } \mathbb{R}^N \tag{5.2}$$

has a unique weak solution in $D^{1,2}(\mathbb{R}^N)$.

Proof. It follows by the Hardy inequality and the coercivity of the functional

$$J(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{2} \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} v^2\,dx - \int_{\mathbb{R}^N} f v\,dx.$$

$\square$

Next we state the Pohozaev identity for a weak solution of

$$-\Delta u = f(x, u) \qquad \text{in } \mathbb{R}^N \tag{5.3}$$

Lemma 5.3. *Let $u \in D^{1,2}(\mathbb{R}^N)$ be a weak solution of (5.3) and let $F(x,u) = \int_0^u f(x,t)\,dt$. Assume furthermore that $u \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and that $F(x,u), x \cdot F_x(x,u) \in L^1(\mathbb{R}^N)$, where F_x is the gradient of F with respect to x. Then u satisfies*

$$\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{2N}{N-2} \int_{\mathbb{R}^N} F(x,u) - \frac{2}{N-2} \int_{\mathbb{R}^N} x \cdot F_x(x,u) = 0. \tag{5.4}$$

Proof. We can proceed exactly as in the proof of Proposition 1 of [BL]. There are only two differences: one is the presence of the term $x \cdot F_x(x,u)$ and the second one is that the solution $u \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and so we have to integrate (5.3) in $B_R \setminus B_\rho$. Anyway these terms can be handled exactly as in the proof of [BL]. □

Here we prove some results that deal with the infimum (1.31) and some other related results in the same spirit of what we proved in the Section 2 of [GGN2].

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$. Moreover assume that*

$$\nu_1 = \inf_{\substack{\eta \in H^1_0(\Omega) \\ \eta \not\equiv 0}} \frac{\int_\Omega |\nabla \eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta^2}{|x|^2}} < 0 \tag{5.5}$$

with $a(x) \in L^\infty(\Omega)$. Then ν_1 is achieved. The function $\psi_1 \in H^1_0(\Omega)$ that achieves ν_1 is strictly positive in $\Omega \setminus \{0\}$ satisfies

$$\int_\Omega \nabla \psi_1 \cdot \nabla \phi - a(x)\psi_1 \phi \, dx = \nu_1 \int_\Omega \frac{\psi_1 \phi}{|x|^2}\, dx \tag{5.6}$$

for any $\phi \in H^1_0(\Omega)$, and the eigenvalue ν_1 is simple.

Proof. Let us consider a minimizing sequence $\eta_n \in H^1_0(\Omega)$ for ν_1, i.e.,

$$\frac{\int_\Omega |\nabla \eta_n|^2 - \int_\Omega a(x)\eta_n^2}{\int_\Omega \frac{\eta_n^2}{|x|^2}} = \nu_1 + o(1). \tag{5.7}$$

Let us normalize η_n such that

$$\int_\Omega \eta_n^2 = 1. \tag{5.8}$$

Then, since $\nu_1 < 0$, by (5.7) we get

$$\int_\Omega |\nabla \eta_n|^2 - \int_\Omega a(x)\eta_n^2 \le 0 \tag{5.9}$$

and then, since a is bounded and (5.8) we deduce from (5.9) that

$$\int_\Omega |\nabla \eta_n|^2 \le C \int_\Omega \eta_n^2 \le C. \tag{5.10}$$

Hence $\eta_n \rightharpoonup \eta$ weakly in $H_0^1(\Omega)$ and then it holds,

$$\int_\Omega |\nabla\eta|^2 \leq \liminf_{n\to+\infty} \int_\Omega |\nabla\eta_n|^2 \tag{5.11}$$

$$\int_\Omega a(x)\eta_n^2 \to \int_\Omega a(x)\eta^2. \tag{5.12}$$

So we get

$$\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2 \leq \liminf_{n\to+\infty} \int_\Omega |\nabla\eta_n|^2 - \int_\Omega a(x)\eta_n^2 + o(1) \tag{5.13}$$

which implies, since $\nu_1 < 0$, and $1 = \int_\Omega \eta_n^2 \leq C \int_\Omega \frac{\eta_n^2}{|x|^2}$,

$$\frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\limsup\limits_{n\to+\infty} \int_\Omega \frac{\eta_n^2}{|x|^2}} \leq \nu_1. \tag{5.14}$$

Then elementary properties of lim inf and lim sup imply

$$\limsup_{n\to+\infty} \frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta_n^2}{|x|^2}} \leq \nu_1. \tag{5.15}$$

Moreover, by Hardy's inequality

$$\int_\Omega \frac{\eta_n^2}{|x|^2} \leq \frac{(N-2)^2}{4} \int_\Omega |\nabla\eta_n|^2 \leq C \tag{5.16}$$

and so, by semicontinuity,

$$\int_\Omega \frac{\eta^2}{|x|^2} \leq \liminf_{n\to+\infty} \int_\Omega \frac{\eta_n^2}{|x|^2}. \tag{5.17}$$

Hence, again using that $\nu_1 < 0$, we get from (5.14) that

$$\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2 < 0. \tag{5.18}$$

On the other hand, from (5.17) we get

$$\limsup_{n\to+\infty} \frac{1}{\int_\Omega \frac{\eta_n^2}{|x|^2}} = \frac{1}{\liminf\limits_{n\to+\infty} \int_\Omega \frac{\eta_n^2}{|x|^2}} \leq \frac{1}{\int_\Omega \frac{\eta^2}{|x|^2}} \tag{5.19}$$

and then

$$\frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta^2}{|x|^2}} \leq \liminf_{n\to+\infty} \frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta_n^2}{|x|^2}}. \tag{5.20}$$

Finally by (5.14) we get

$$\frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta^2}{|x|^2}} \le \nu_1 \tag{5.21}$$

which proves the first part of the proposition. The rest follows exactly as in proof of Proposition 2.1 of [GGN2] □

The same result holds also if we minimize the quadratic form (5.5) with some orthogonality conditions. To this end we say that ψ and η are orthogonal if they satisfy $\int_\Omega \frac{\psi\eta}{|x|^2}\,dx = 0$. Indeed we have the following:

Proposition 5.5. *Let us assume Ω, ν_1, ψ_1 and $a(x)$ as in Proposition 5.4. Then if we have that*

$$\nu_2 = \inf_{\substack{\eta\in H^1_0(\Omega),\\ \eta\perp\psi_1}} \frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta^2}{|x|^2}} < 0 \tag{5.22}$$

then ν_2 is achieved. Moreover the functions $\psi_2 \in H^1_0(\Omega)$ that attains ν_2 satisfies

$$\int_\Omega \nabla\psi_2\cdot\nabla\phi - a(x)\psi_2\phi\,dx = \nu_2\int_\Omega \frac{\psi_2\phi}{|x|^2}\,dx$$

for any $\phi\in H^1_0(\Omega)$.

Similarly for $i = 3, .., k$, if we have that

$$\nu_i = \inf_{\substack{\eta\in H^1_0(\Omega),\\ \eta\perp span\{\psi_1,\psi_2,..,\psi_{i-1}\}}} \frac{\int_\Omega |\nabla\eta|^2 - \int_\Omega a(x)\eta^2}{\int_\Omega \frac{\eta^2}{|x|^2}} < 0 \tag{5.23}$$

then ν_i is achieved and the functions $\psi_i\in H^1_0(\Omega)$ that attain ν_i satisfy

$$\int_\Omega \nabla\psi_i\cdot\nabla\phi - a(x)\psi_i\phi\,dx = \nu_i\int_\Omega \frac{\psi_i\phi}{|x|^2}\,dx$$

for any $\phi\in H^1_0(\Omega)$.

Proof. It is the same of the previous lemma. For any i let us consider a minimizing sequence $\eta_{i,n}\in H^1_0(\Omega)$ for ν_i . Then it converges to a function η_i which achieves ν_i and that is a weak solution of the equation. □

Now we use the previous result to compute the Morse index of the radial solution u_λ to (1.1). We state the result in the case of $\Omega = B_1$.

Lemma 5.6. *Let u_λ be a solution to (1.28) whose Morse index is $M > 0$. Then there exist exactly M functions $\psi_i\in H^1_0(B_1)$ and M numbers $\nu_i < 0$ such that the problem*

$$\begin{cases} -\Delta\psi_i - \frac{\lambda}{|x|^2}\psi_i - pu_\lambda^{p-1}\psi_i = \frac{\nu_i}{|x|^2}\psi_i, \ \ in \ \ B_1\setminus\{0\}\\ \psi_i\in H^1_0(B_1)\end{cases} \tag{5.24}$$

admits a weak solution. The functions ψ_i can be taken in such a way they verify

$$\int_{B_1} \frac{\psi_i \psi_j}{|x|^2}\, dx = 0 \quad \textit{for } i \neq j. \tag{5.25}$$

The proof follows exactly as in the proof of Lemma 2.6 in [GGN2] and we do not report it.
The results of Propositions 5.4 and 5.5 and Lemma 5.6 hold true also if we let $\Omega = \mathbb{R}^N$ and substitute $H_0^1(\Omega)$ with $D^{1,2}(\mathbb{R}^N)$. Then we can state the following:

Corollary 5.7. *The Morse index of the radial solution u_λ of (1.16) is given by the number of negative values Λ_i such that the problem*

$$\begin{cases} -\Delta w - \frac{\lambda}{|x|^2} w - N(N+2)\nu_\lambda^2 \frac{|x|^{2\nu_\lambda}}{\left(1+|x|^{2\nu_\lambda}\right)^2} w = \frac{\Lambda_i}{|x|^2} w & \textit{in } \mathbb{R}^N \\ w \in D^{1,2}\left(\mathbb{R}^N\right) \end{cases} \tag{5.26}$$

admits a weak solution, counted with their multiplicity.

Proof. Let u_λ be the radial solution of (1.16). Then we can use the analouguos of Lemma 5.6 in $\mathbb{R}^N$ to prove the claim. □

Finally the result of Proposition 5.4 can be used also to prove that the first eigenvalue with weight (1.31) is attained. Indeed we have:

Proposition 5.8. *Assume that*

$$\Lambda_1 = \inf_{\substack{\eta \in H^1((0,1), r^{M-1}dr), \eta(1)=0 \\ \eta \not\equiv 0}} \frac{\int_0^1 r^{M-1} |\eta'|^2\, dr - \int_0^1 r^{M-1} a(r) \eta^2\, dr}{\int_0^1 r^{M-3} \eta^2\, dr} < 0. \tag{5.27}$$

with $a \in L^\infty(0,1)$. Then Λ_1 is achieved.

Proof. The claim follows as in the proof of Proposition 5.4. □

Lemma 5.9. *Let us consider a solution to*

$$\begin{cases} -\psi'' - \frac{M-1}{r}\psi' + \beta^2 \frac{\psi}{r^2} = h\psi, & \textit{in } (0,1) \\ \psi(1) = 0, \ \int_0^1 r^{M-1} (\psi')^2 dr < \infty \end{cases} \tag{5.28}$$

with $h \in L^\infty(0,1)$ and $\beta \neq 0$. Then $\psi \in L^\infty(0,1)$ and $\psi(0) = 0$.

Proof. Let $\theta = \frac{2-M+\sqrt{(M-2)^2+4\beta^2}}{2} > 0$. Since $\int_0^1 r^{M-1}(\psi')^2 dr < +\infty$ we get by (5.28) that $\int_0^1 \psi^2 r^{M-3} dr < +\infty$. Then there exists a sequence $r_n \to 0$ such that $r_n^{\theta+M-2}\psi(r_n) = o(1)$ as $n \to +\infty$ for any $\beta > 0$. Such a sequence exists because, if not, we get $\psi(r) \geq \frac{C}{r_n^{\theta+M-2}}$ in a suitable neighborhood of 0 and this contradicts that $\int_0^1 \psi^2 r^{M-3} dr < +\infty$ (note that we have used that

$\theta > \frac{2-M}{2}$).
Let us observe that the function $v(r) = r^\theta$ satisfies

$$\begin{cases} -v'' - \frac{M-1}{r}v' + \frac{\beta^2}{r^2}v = 0, & \text{in } (0,+\infty) \\ v(0) = 0. \end{cases} \tag{5.29}$$

From (5.29) and (5.28) we obtain, integrating on (r_n, R),

$$\int_{r_n}^{R} s^{\theta+M-1}h(s)\psi(s)ds = -R^{\theta+M-1}\psi'(R)+r_n^{\theta+M-1}\psi'(r_n)+\theta R^{\theta+M-2}\psi(R)-\theta r_n^{\theta+M-2}\psi(r_n) \tag{5.30}$$

We claim that

$$r_n^{\theta+M-1}\psi'(r_n) = o(1)\ , \quad \text{as } n\to\infty \tag{5.31}$$

Integrating (5.28) we get

$$r_n^{\theta+M-1}\psi'(r_n) = O\left(r_n^\theta\right)+r_n^\theta\int_{r_n}^{1} s^{M-1}h(s)\psi(s)ds-\beta^2 r_n^\theta\int_{r_n}^{1} s^{M-3}\psi(s)ds = o(1)$$

since $r_n^\theta\int_{r_n}^1 s^{M-3}\psi(s) \le r_n^\theta\left(\int_{r_n}^1 \frac{\psi^2(s)}{s^2}s^{M-1}\right)^{\frac12}\left(\int_{r_n}^1 s^{M-3}\right)^{\frac12} = o(1)$ and this proves (5.31). Hence (5.30) becomes

$$\int_0^R s^{\theta+M-1}h(s)\psi(s)ds = -R^{\theta+M-1}\psi'(R) + \theta R^{\theta+M-2}\psi(R) \tag{5.32}$$

Then we deduce that

$$\frac{\psi(t)}{t^\theta} = \int_t^1 \frac{1}{R^{2\theta+M-1}}\left(\int_0^R s^{\theta+M-1}h(s)\psi(s)ds\right)dR. \tag{5.33}$$

Now, since $h\in L^\infty(0,1)$ we get

$$\left|\int_0^R s^{\theta+M-1}h(s)\psi(s)\right| \le C\int_0^R s^{\theta+\frac{M+1}{2}}\psi(s)s^{\frac{M-3}{2}} \le C\left(\int_0^R s^{2\theta+M+1}\right)^{\frac12}\left(\int_0^R \psi^2(s)s^{M-3}\right)^{\frac12}$$
$$\le CR^{\theta+\frac{M+2}{2}}. \tag{5.34}$$

Finally (5.33) becomes

$$\psi(t) = \begin{cases} O\left(t^\theta\right) & \text{if } \theta < -\frac{M}{2}+3 \\ O\left(t^{-\frac{M}{2}+3}\right) & \text{if } \theta > -\frac{M}{2}+3 \\ O\left(t^{-\frac{M}{2}+3}|\log t|\right) & \text{if } \theta = -\frac{M}{2}+3 \end{cases} \tag{5.35}$$

Then if $M < 6$ (5.35) implies that $\psi(0) = 0$ which gives the claim. If $M \geq 6$, instead, we have $-\frac{M}{2} + 3 \leq 0 < \theta$ and so $\psi(t) = O\left(t^{-\frac{M}{2}+3}\right)$. Plugging this estimate in (5.34) we get

$$\left|\int_0^R s^{\theta+M-1} h(s)\psi(s)\right| \leq CR^{\theta+\frac{M+6}{2}}. \tag{5.36}$$

and (5.33) becomes

$$\psi(t) = \begin{cases} O\left(t^{\theta}\right) & \text{if } \theta < -\frac{M}{2} + 5 \\ O\left(t^{-\frac{M}{2}+5}\right) & \text{if } \theta \neq -\frac{M}{2} + 5 \\ O\left(t^{-\frac{M}{2}+5}|\log t|\right) & \text{if } \theta = -\frac{M}{2} + 5 \end{cases} \tag{5.37}$$

which gives the claim for $M < 10$. Iterating the procedure in a finite number of steps we get that $\psi(t) = O\left(t^{\theta}\right)$ as $t \to 0$ so that $\psi(0) = 0$. This ends the proof. □

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